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# Some Properties of Large $N$
# Two Dimensional Yang–Mills Theory

David J. Gross[†]

gross@puhep1.princeton.edu

and

Andrei Matytsin

matytsin@puhep1.princeton.edu

*Department of Physics*
*Joseph Henry Laboratories*
*Princeton University*
*Princeton, NJ  08544*

## Abstract

Large $N$ two-dimensional QCD on a cylinder and on a vertex manifold (a sphere with three holes) is investigated. The relation between the saddle-point description and the collective field theory of $QCD_2$ is established. Using this relation, it is shown that the Douglas–Kazakov phase transition on a cylinder is associated with the presence of a gap in the eigenvalue distributions for Wilson loops. An exact formula for the phase transition on disc with an arbitrary boundary holonomy is found. The role of instantons in inducing such transitions is discussed. The zero-area limit of the partition function on a vertex manifold is studied. It is found that this partition function vanishes unless the boundary conditions satisfy a certain selection rule which is an analogue of momentum conservation in field theory.

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## 1. Introduction.

Recently there has been much interest in the study of $QCD_2$ in the large $N$ limit, largely motivated by an attempt to find a string representation of QCD in four dimensions [1]. The study of large $N$ $QCD_2$ is also useful in exploring more genral properties of large $N$ QCD. As such, a detailed analysis of the properties of the theory on the sphere and on the cylinder are interesting. Perhaps the most important of these properties is the third order phase transition for large $N$ QCD on a sphere, discovered recently by Douglas and Kazakov [2]. Physically, this transition is caused by condensation of instantons. One can demonstrate that the effect of instantons is negligible when the area of the sphere is small, and becomes significant as the area reaches the critical value, $A_{cr} = \pi^2/\lambda$, where $\lambda = g^2 N$ is the large $N$ coupling constant of QCD [3].

The existence of this phase tranition, which is similar to the phase transition that occurs in the one-plaquette model [4], is of great interest. Such effects signal a sharp transition between weak and strong coupling, between non-confining physics and confinement, and between the a stringy regime and a non-string regime. If they were to occur in four dimensional QCD they might be an indication that the string picture is limited to infrared phenomena and cannot be analytically continued to the ultraviolet domain. One of our goals is to analyse the occurence of such phase transitions for other geometries. We will develop methods that allow us to explore this phase transition in detail.

A phase transition similar to the Douglas-Kazakov transition occurs in $QCD_2$ on a cylinder with fixed boundary conditions (that is, with fixed holonomy matrices $U_{C_1} = \mathcal{P} \exp \oint_{C_1} \mathcal{A}_\mu(x) \, dx^\mu$ and $U_{C_2} = \mathcal{P} \exp \oint_{C_2} \mathcal{A}_\mu(x) \, dx^\mu$. Here $C_1$ and $C_2$ are the two circles forming the boundary of the cylinder). Obviously, such a phase transition should manifest itself in the master field of two-dimensional QCD [5].

A natural set of physical observables in gauge theory is formed by the Wilson loops $W_n(C) = \frac{1}{N} \mathrm{Tr}\, U_C^n$ with $U_C = \mathcal{P} \exp \oint_C \mathcal{A}_\mu(x) \, dx^\mu$. Therefore, we could say that the master field is fully described by the set of quantities $W_n(C)$ for all possible contours $C$. Indeed, $W_n(C)$ satisfy a closed set of equations (the loop equations) with a well defined large $N$ limit [6][7][8]. In addition, the expectation values of Wilson loops do factorize at large $N$. The loop equations, however, are higly nonlocal and the description of the theory they provide is very convoluted.

The loop variables are difficult to work with. For two-dimensional QCD a better set of variables is readily available. Indeed, it is known that the two-dimensional QCD is equivalent to a $c = 1$ matrix model with the spatial coordinate compactified on a circle [9][10][11]. Therefore, the eigenvalue density of the Wilson matrix $U_C$ will satisfy the appropriate collective field equation (the Hopf equation) as a function of the area bounded

by the contour $C$. Using this fact we can obtain a formula for Wilson loops $W_n$ in the large $N$ QCD on a cylinder with arbitrary boundary conditions.

As a result, the dynamics of Wilson loops on a cylinder can be desribed by a simple physical picture. Let $\sigma_1(\theta)$ and $\sigma_2(\theta)$, $\theta \in [0, 2\pi]$, be the eigenvalue densities of the boundary holonomy matrices $U_{C_1}$ and $U_{C_2}$. Consider a one-dimensional compressible fluid living on a circle, with the equation of state $P = -\frac{\pi^2}{2}\sigma^2$, $P$ being the pressure and $\sigma$ the density. Imagine the process where the fluid moves from an initial configuration, where its density profile is $\sigma(\theta) = \sigma_1(\theta)$ to a final one, $\sigma(\theta) = \sigma_2(\theta)$, during the time interval equal to the total area of the cylinder, $A$. Then the eigenvalue density for the Wilson matrix $U_C$, where the contour $C$ cuts the cylinder of total area $A$ into two cylinders of areas $A_1$ and $A_2$, is just given by the density of our fluid at the time $A_1$. Consequently,

$$W_n(C) = \int_0^{2\pi} \sigma\left(t = A_1, \theta\right) e^{in\theta} \, d\theta. \tag{1.1}$$

On the other hand, the partition function of the two-dimensional QCD, at least in the most interesting case of symmetric boundary conditions $U_{C_1} = U_{C_2}^\dagger$, is dominated by a single representation of the gauge group. This representation, associated with its Young tableau, can be characterized by a set of numbers $h_i = l_i/N$, $i = 1, \ldots, N$. The density of this set, $\rho_Y(h)$, plays the crucial role in the analysis of the Douglas-Kazakov phase transition. However, there is no known equation for determining $\rho_Y(h)$, except for some special cases.
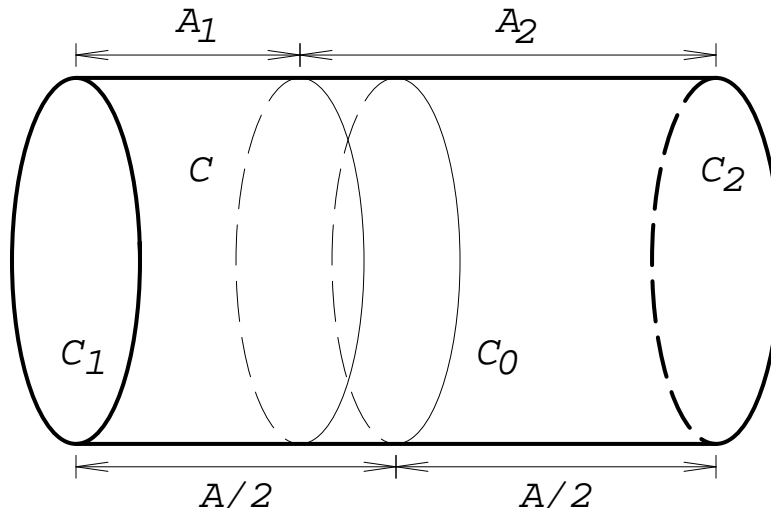


**Fig. 1:** The cylinder with boundary contours $C_1$ and $C_2$. The contour $C_0$, used in the calculation of the dominant Young tableau, cuts the cylinder in half.

This problem shall be solved below, in section 2. We find that, when $U_{C_1} = U_{C_2}^\dagger$, the Young tableau density $\rho_Y(h)$ is determined by the surprisingly simple formula

$$\pi\rho_Y\big(-\pi\sigma_0(\theta)\big) = \theta \tag{1.2}$$

where $\sigma_0(\theta)$ is the density of eigenvalues for the Wilson matrix $U_{C_0}$. Here the contour $C_0$ cuts the cylinder into two equal parts, $A_1 = A_2 = A/2$ (see fig. 1). The density $\sigma_0(\theta)$ can be found using the procedure outlined above[1]. This formula will also allow us to prove the criterion (formulated by Caselle, D'Adda, Magnea and Panzeri [11]) determining for which boundary conditions the partition function of $QCD_2$ can exhibit a Douglas-Kazakov phase transition, and for which the transition is absent. This criterion applies even in those situations when no single representation is dominant and the original method of Douglas and Kazakov leads to a complex saddle point. Quite remarkably, we discover that for QCD on a disc (that is, when $\sigma_2(\theta) = \delta(\theta)$) the transition point can be determined exactly and explicitly to be

$$A_{\mathrm{cr}} = \pi\left[\int \frac{\sigma_1(\theta)\,d\theta}{\pi - \theta}\right]^{-1}.$$

As a second application of formula (1.2) we will consider the well studied example of QCD on a plane. We find that the structure of Wilson loops $W_n(A)$ with large values of winding number $n$ experiences qualitative changes as the area enclosed by the loop, $A$, passes through the critical value $A_{\mathrm{pl}} = 4$. This phenomenon, which also can be viewed as a phase transition, is in fact a remnant of the Douglas-Kazakov transition occuring on a sphere of finite area $A = \pi^2$.

We will use this result in section 3 to make some general comments about the implications and meaning of the phase transition.

The collective field theory approach to $QCD_2$ is very powerful, at least on the cylinder. The Hopf action, as we shall discuss, propagates Wilson loops, or eigenvalue densities, along the cylinder.

To discuss other geometries, in particular surfaces with handles, we have to know how to split the loops at a vertex. To this end we need the partition function for $QCD_2$ on a "pair of trousers"–type manifold (fig. 2) (that is, a sphere with three holes). Moreover, it suffices to consider the limiting case when the area of this manifold vanishes. Indeed, we can always create the "pair of trousers" with a finite area by attaching cylinders to the zero-area "vertex".

The properties of these string vertices shall be investigated in section 4. We find that, as $N \to \infty$, the partition function of the QCD on a "vertex" vanishes in the zero-area limit, unless certain selection rule is satisfied. We find and prove this selection rule and discuss its implications for the string theory of two-dimensional QCD.

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[1] See the discussion preceding (1.1).
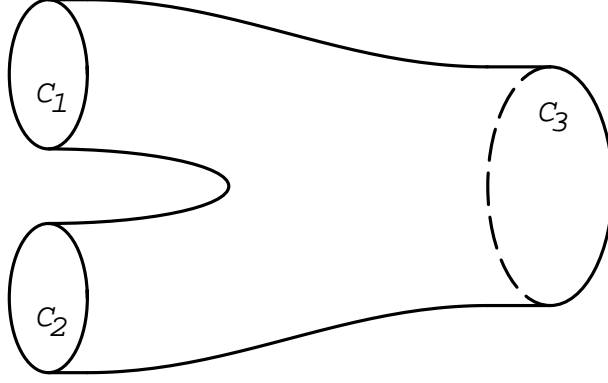
**Fig. 2:** The vertex manifold with three boundaries, $C_1$, $C_2$ and $C_3$.

## 2. Large $N$ QCD on a cylinder.

The partition function of QCD on a cylinder (fig.1) can be evaluated exactly for any finite $N$ and equals [12][13]

$$\mathcal{Z}_N(U_{C_1}, U_{C_2}|A) = \sum_R \chi_R(U_{C_1})\chi_R(U_{C_2}^\dagger)\, e^{-\frac{\lambda A}{2N}C_2(R)} \tag{2.1}$$

where $\lambda = g^2 N$ is the large $N$ coupling constant and $A$ the area of the cylinder. The summation is over all irreducible representations $R$ of the gauge group, $\chi_R(U)$ is the character of the matrix $U$ in the representation $R$, and $C_2(R)$ is the quadratic Casimir operator of this representation. Since the answer depends only on the product of $\lambda$ and $A$, we shall set $\lambda = 1$.

If the gauge group is $U(N)$, the representations $R$ can be labelled by a set of integers $+\infty > l_1 > l_2 > \ldots > l_N > -\infty$. Then

$$\chi_R(U) = \frac{\det\left\|e^{il_j\theta_k}\right\|}{J\left(e^{i\theta_s}\right)} \tag{2.2}$$

where $e^{i\theta_k}$; $k = 1,\ldots,N$ are the eigenvalues of the (unitary) matrix $U$, and

$$J\left(e^{i\theta_s}\right) = \prod_{j<k}\left(e^{i\theta_j} - e^{i\theta_k}\right).$$

Also,

$$C_2(R) = \frac{N}{12}(N^2 - 1) + \sum_{i=1}^{N}\left(l_i - \frac{N-1}{2}\right)^2.$$

4

In this section we will study the partition function (2.1) in the large $N$ limit. This can be done in two ways. One way is to show that for large $N$ (2.1) is dominated by a particular representation $R$. If this representation is found, it is possible to express all physical observables in terms of the indices $\{l_i\}$ labelling $R$. The second approach is to construct an appropriate collective field theory describing $QCD_2$. While the latter method is more physically transparent, practical calculations have so far been carried out using the first approach. For example, it is not at all obvious how to find the solution of the collective field theory which describes the strong coupling phase of QCD on a sphere.

These two approaches are complementary. Below we will derive the equation (1.2) relating the dominant representation to the solution of the collective theory. This will allow us to provide a simple physical interpretation of the formulas for Wilson loops on a sphere, found recently by Boulatov, Daul and Kazakov and extend them to the general case of a cylinder.

### 2.1. The dominant representation.

Generally, infinitely many representations contribute to the partition function (2.1). However, as $N \to \infty$ the sum is dominated by a single representation. Indeed, for large $N$ the $U(N)$ characters (2.2) behave asymptotically as

$$\chi_R(U) \simeq e^{N^2 \Xi[\rho_Y(l/N), \sigma(\theta)]} \tag{2.3}$$

with some finite functional $\Xi[\rho_Y, \sigma]$. In this formula it is implicit that we take the limit $N \to \infty$ assuming that the eigenvalue distribution of the unitary $N \times N$ matrix $U$ converges to a smooth function[2] $\sigma(\theta)$, $\theta \in [0, 2\pi]$. In addition, it is assumed that the distribution of parameters $\tilde{y}_i = l_i/N$, which define the representation $R$, also converges to another smooth function $\rho_Y(\tilde{y})$, that we can call the Young tableau density. The functional $\Xi$ is, in general, not easy to calculate. However, in some important cases it can be found explicitly.

As a result,

$$
\begin{aligned}
\mathcal{Z}_N(U_{C_1}, U_{C_2}|A) \simeq \sum_R \exp\Bigg[ N^2 \Big\{ &\Xi\big[\rho_Y(\tilde{y}), \sigma_1(\theta)\big] + \overline{\Xi\big[\rho_Y(\tilde{y}), \sigma_2(\theta)\big]} \\
&- \frac{A}{2} \int \rho_Y(\tilde{y})\Big(\tilde{y} - \frac{1}{2}\Big)^2 d\tilde{y} - \frac{A}{24} \Big\} \Bigg],
\end{aligned}
$$

---

[2] The eigenvalues of a unitary matrix lie on the unit circle in the complex plane and can be parametrized as $\lambda_j = e^{i\theta_j}$.

where the bar denotes complex conjugation. In this formula the summation over all representations $R$ can be thought of as "functional integration" over all possible distributions of the Young tableaux, $\rho_Y(\tilde{y})$.

For large $N$ we expect that this "functional integral" is dominated by a saddle point, which can be determined from the equation

$$\frac{\partial}{\partial \tilde{y}}\left[\frac{\delta \Xi\big[\rho_Y(\tilde{y}), \sigma_1(\theta)\big]}{\delta \rho_Y(\tilde{y})} + \frac{\delta \overline{\Xi\big[\rho_Y(\tilde{y}), \sigma_2(\theta)\big]}}{\delta \rho_Y(\tilde{y})}\right] = A\Big(\tilde{y} - \frac{1}{2}\Big). \tag{2.4}$$

Then the free energy of the theory equals

$$F_N(U_{C_1}, U_{C_2}|A) \equiv \ln \mathcal{Z}_N(U_{C_1}, U_{C_2}|A) = N^2\Big[\Xi\big[\rho_Y(y), \sigma_1(\theta)\big]$$
$$+ \overline{\Xi\big[\rho_Y(y), \sigma_2(\theta)\big]} - \frac{A}{2}\int \rho_Y(y)\, y^2 dy - \frac{A}{24}\Big] + \mathcal{O}(N^0)$$

where to simplify the notation we have used a shifted variable $y = \tilde{y} - 1/2$. It is also useful to keep in mind that the derivative of $F_N$ with respect to the area $A$ (the "specific heat capacity") is given by[3]

$$\frac{1}{N^2}\frac{\partial F_N}{\partial A} = -\frac{1}{24} - \frac{1}{2}\int \rho_Y(y)\, y^2 \, dy. \tag{2.5}$$

Since the indices $l_i$ are discrete integers, the differences $l_i - l_{i+1}$ are always greater than one, and therefore the density $\rho_Y(y)$ must satisfy the constraint $\rho_Y(y) \leq 1$. The saturation of this bound causes a phase transition for large $N$ QCD [2].

To see how this phase transition occurs, let us consider the two-dimensional QCD on a sphere. We can regard the sphere as a particular case of a cylinder with boundary conditions $U_{C_1} = U_{C_2} = \mathrm{I}$ (that is, $\sigma_1(\theta) = \sigma_2(\theta) = \delta(\theta)$). In this situation the large $N$ limit of characters is easy to calculate. Indeed, since

$$\chi_R(\mathrm{I}) = d_R = \prod_{i<j}(l_i - l_j),$$

we have

$$\Xi\big[\rho_Y(y), \delta(\theta)\big] = \frac{1}{2}\int \rho_Y(y_1)\, \rho_Y(y_2)\, \ln|y_1 - y_2|\, dy_1\, dy_2\,.$$
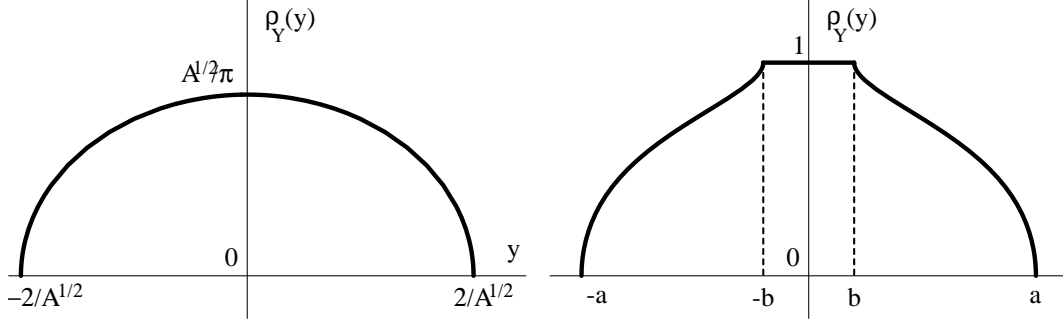
Then (2.4) entails

$$\int \frac{\rho_Y(u)\, du}{y - u} = \frac{Ay}{2}.$$

The solution of this equation is

$$\rho_Y(y) = \frac{1}{\pi}\sqrt{A - \frac{A^2 y^2}{4}}. \tag{2.6}$$

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[3] Although $\rho_Y(y)$ by itself depends on the area, this dependence does not contribute to the specific heat due to the saddle point equation (2.4).

**Fig. 3:** The density $\rho_Y(y)$ describing the dominant Young tableau for a sphere in the weak coupling phase (left) and in the strong coupling phase (right).

However, it obeys the constraint $\rho_Y(y) \le 1$ only if $A \le \pi^2$. If $A > \pi^2$ then, instead, we must set $\rho_Y(y) = 1$ in some interval $y \in [-b, b]$ and the solution becomes [2]

$$\rho_Y(y) = \frac{2}{\pi a y} \sqrt{(a^2 - y^2)(y^2 - b^2)} \, \Pi_1\left(-\frac{b^2}{y^2}, \frac{b}{a}\right) \tag{2.7}$$

outside of this interval, for $y \in [b, a]$. In this formula

$$\Pi_1(x, k) = \frac{1}{2} \int_{-1}^{+1} \frac{du}{1 + xu^2} \frac{1}{\sqrt{(1 - k^2 u^2)(1 - u^2)}} \tag{2.8}$$

is the elliptic integral of the third kind and the constants $b$ and $a$ are fixed by

$$\begin{cases} k = b/a, \qquad k' = \sqrt{1 - k^2}, \\ a\left(2\mathbf{E}(k) - k'^2 \mathbf{K}(k)\right) = 1, \\ aA = 4\mathbf{K}(k). \end{cases} \tag{2.9}$$

Accordingly, the specific heat (2.5) is different for $A < \pi^2$ and $A > \pi^2$. In fact, it is possible to show that at the critical point $A = \pi^2$ the third derivative of the free energy is discontinuous, indicating a phase transition.

Obviously, a similar transition will occur on the cylinder. In this case the saddle point equation (2.4) becomes very involved. To find its solution we will have to use an entirely different treatment of $\mathrm{QCD}_2$, based on the ideas of the collective field theory.

### 2.2. The collective field theory.

The large $N$ limit of two-dimensional QCD can also be studied with the aid of a collective field theory. The underlying idea of collective field methods is to make a change

of variables describing the theory so that the new variables (the "collective variables") would have a well defined large $N$ limit [14].

In our case the partition function $\mathcal{Z}_N$ depends on the two sets of eigenvalues $\{\lambda_a^{(1)} = e^{i\theta_a^{(1)}} | a = 1, \ldots, N\}$ and $\{\lambda_b^{(2)} = e^{i\theta_b^{(2)}} | b = 1, \ldots, N\}$. The specific eigenvalues do not tend to any limit as we take $N$ to infinity. However, the corresponding eigenvalue distributions $\sigma_1(\theta)$ and $\sigma_2(\theta)$ do have a large $N$ limit. In fact, $\sigma_1$ and $\sigma_2$ contain all the data one needs to evaluate the leading $N \to \infty$ asymptotics of $\mathcal{Z}_N$.

Therefore, it must be possible to obtain a differential equation on $\mathcal{Z}_N$ as a functional of $\sigma_1$ and $\sigma_2$. To achieve this goal we will write down a differential equation satisfied by $\mathcal{Z}_N$ as a function of the $2N$ discrete variables $\{\theta_a^{(1)}\}$ and $\{\theta_b^{(2)}\}$. Then we will change variables to $\sigma_1(\theta)$ and $\sigma_2(\theta)$.

As a first step, one represents (2.1) in the form

$$
\mathcal{Z}_N(U_{C_1}, U_{C_2}|A) = \frac{1}{N!} \frac{1}{2^{N(N-1)}}
$$

$$
\times \sum_{y_k \in \{\pm\frac{1}{N}, \pm\frac{2}{N}, \ldots\}} \frac{\det \left\| e^{iy_a \theta_b^{(1)}} \right\| \det \left\| e^{-iy_c \theta_d^{(2)}} \right\|}{\prod\limits_{s<r} \left[ \sin \frac{\theta_s^{(1)} - \theta_r^{(1)}}{2} \sin \frac{\theta_s^{(2)} - \theta_r^{(2)}}{2} \right]} e^{-\frac{A}{2}N \sum\limits_{k=1}^{N} y_k^2} \tag{2.10}
$$

with $y_k = \frac{1}{N}\left(l_k - \frac{N-1}{2}\right)$. It is now obvious that $\tilde{\mathcal{Z}}_N \equiv \mathcal{Z}_N \, e^{\frac{A}{24}(N^2-1)}$ satisfies the differential equation

$$
2N \frac{\partial}{\partial A} \tilde{\mathcal{Z}}_N = \frac{1}{\mathcal{D}(\theta^{(1)})} \sum_{k=1}^{N} \frac{\partial^2}{\partial \theta_k^{(1)2}} \left[ \mathcal{D}(\theta^{(1)}) \tilde{\mathcal{Z}}_N \right] \tag{2.11}
$$

where

$$
\mathcal{D}(\theta^{(1)}) \equiv \prod_{s<r} \sin \frac{\theta_s^{(1)} - \theta_r^{(1)}}{2}.
$$

If $\tilde{\mathcal{Z}}_N = \exp N^2 \tilde{F}_N$, then $\tilde{F}_N$ has a well defined large $N$ limit $\tilde{F} = \lim\limits_{N \to \infty} \tilde{F}_N$. The limiting functional $\tilde{F}$ can be determined if we convert (2.11) into an equation for $\tilde{F}_N$ and then change variables from $\{\theta_a^{(1)}\}$ to $\sigma_1(\theta)$ in that equation. As a final result of these manipulations, it follows that[4]

$$
\tilde{F}\left[\sigma_1(\theta), \sigma_2(\theta)\big|A\right] = S\left[\sigma_1(\theta), \sigma_2(\theta)\big|A\right]
$$

$$
-\frac{1}{2}\int \sigma_1(\theta)\sigma_1(\varphi) \ln \left| \sin \frac{\theta - \varphi}{2} \right| d\theta \, d\varphi - \frac{1}{2}\int \sigma_2(\theta)\sigma_2(\varphi) \ln \left| \sin \frac{\theta - \varphi}{2} \right| d\theta \, d\varphi \tag{2.12}
$$

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[4] The procedure of changing variables, which is well known but nontrivial, is described in Appendix A.

where the functional $S$ is a solution of

$$\frac{\partial S}{\partial A} = \frac{1}{2} \int\limits_0^{2\pi} \sigma_1(\theta) \left[ \left( \frac{\partial}{\partial \theta} \frac{\delta S}{\delta \sigma_1(\theta)} \right)^2 - \frac{\pi^2}{3} \sigma_1^2(\theta) \right]. \tag{2.13}$$

It is helpful to interpret (2.13) as the Hamilton–Jacobi equation for the classical Hamiltonian

$$H\big[\sigma(\theta), \Pi(\theta)\big] = \frac{1}{2} \int\limits_0^{2\pi} \sigma(\theta) \left[ \left( \frac{\partial \Pi}{\partial \theta} \right)^2 - \frac{\pi^2}{3} \sigma^2(\theta) \right] \tag{2.14}$$

with $A$ playing the role of time, the function $\sigma_1(\theta)$ being the canonical coordinate and $\Pi(\theta) \equiv \delta S / \delta \sigma_1(\theta)$ the conjugate momentum. The Hamiltonian (2.14) is called the Das–Jevicki Hamiltonian [14].

In fact, the required solution is easy to construct. To do this, we solve the Hamilton equations of motion

$$\frac{\partial \sigma(\theta)}{\partial t} = \frac{\delta H[\sigma, \Pi]}{\delta \Pi(\theta)}, \qquad \frac{\partial \Pi(\theta)}{\partial t} = -\frac{\delta H[\sigma, \Pi]}{\delta \sigma(\theta)} \tag{2.15}$$

and pick the particular solution which satisfies the boundary conditions

$$\begin{cases} \sigma(\theta)\big|_{t=0} = \sigma_1(\theta) \\ \sigma(\theta)\big|_{t=A} = \sigma_2(\theta) \end{cases} \tag{2.16}$$

where, as before, $\sigma_1$ and $\sigma_2$ are the eigenvalue densities of matrices $U_{C_1}$ and $U_{C_2}$. Then it is possible to prove that $S[\sigma_1, \sigma_2|A]$ equals the classical action calculated for this particular solution[5].

The equations of motion which follow from (2.15) are

$$\begin{cases} \dfrac{\partial \sigma}{\partial t} + \dfrac{\partial}{\partial \theta}(\sigma v) = 0 \\ \dfrac{\partial v}{\partial t} + v \dfrac{\partial v}{\partial \theta} = \dfrac{\partial}{\partial \theta} \left( \dfrac{\pi^2}{2} \sigma^2 \right) \end{cases}, \qquad v(\theta) \equiv \frac{\partial \Pi}{\partial \theta}. \tag{2.17}$$

These are the Euler equations for a fluid with negative pressure $P = -\frac{\pi^2}{2}\sigma^2$. The solution we are looking for corresponds to the process where the density profile $\sigma_1(\theta)$ evolves into the profile $\sigma_2(\theta)$ during a time equal to $A$.

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[5] If there are several such solutions, the one with the largest value of $S$ must be chosen.

So far we have seen how this formalism gives us a way of calculating free energy. However, in addition, the collective theory provides a natural and concise description of Wilson loops in the large $N$ QCD.

The application of the collective field theory to the study of Wilson loops is based on the following factorization property. Imagine we cut our cylinder along some contour $C$ (fig. 1) into two pieces of areas $A_1$ and $A_2$. We may ask how the partition function of the whole cylinder is expressed in terms of the partition functions for these two pieces.

The answer to this question is obvious. Using (2.1) along with the orthogonality relation for characters

$$\int dU \, \chi_{R_1}(U)\chi_{R_2}(U^\dagger) = \delta_{R_1,R_2} \tag{2.18}$$

we easily see that

$$\mathcal{Z}_N(U_{C_1}, U_{C_2}|A) = \int dU_C \, \mathcal{Z}_N(U_{C_1}, U_C|A_1) \, \mathcal{Z}_N(U_C, U_{C_2}|A_2). \tag{2.19}$$

That is to say, to glue the two pieces into a single cylinder we have to set their boundary values, at the place we glue, equal to the same matrix $U_C$ and then integrate over all such matrices. If we treat $\mathcal{Z}_N(U_{C_1}, U_{C_2}|A)$ as the probability amplitude for the process where $U_{C_1}$ evolves into $U_{C_2}$ during the time $A$, then (2.19) is just the convolution property satisfied by the Feynman path integrals.

However, in the large $N$ limit the integral in (2.19) is dominated by a single saddle point. Thus the partition function reduces to the product of the partition functions of its two pieces. To see this, recall that the characters $\chi_R(U_C)$ depend only on the eigenvalues of $U_C$, and so does $\mathcal{Z}_N$. Therefore, the integration in (2.19) reduces to[6]

$$\mathcal{Z}_N(\{\theta_a^{(1)}\}, \{\theta_d^{(2)}\}|A)$$
$$= \int d\theta_1^{(C)} \ldots d\theta_N^{(C)} \left[\mathcal{D}(\theta^{(C)})\right]^2 \mathcal{Z}_N(\{\theta_a^{(1)}\}, \{\theta_b^{(C)}\}|A_1) \, \mathcal{Z}_N(\{\theta_b^{(C)}\}, \{\theta_d^{(2)}\}|A_2).$$

Substituting for $\mathcal{Z}_N$ their expressions in terms of $S$ using (2.12), we obtain

$$e^{N^2 S[\sigma_1, \sigma_2|A]} = \int d\theta_1^{(C)} \ldots d\theta_N^{(C)} \, e^{N^2\left(S[\sigma_1, \sigma_C|A_1] + S[\sigma_C, \sigma_2|A_2]\right)}, \tag{2.20}$$

where $\sigma_C(\theta)$ is the distribution of angles $\theta_j^{(C)}$ over which the integration is performed. The integral in (2.20) is indeed dominated by a saddle point. We can even find this saddle

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[6] The factor $\left[\mathcal{D}(\theta^{(C)})\right]^2$ is the jacobian which appears when we express the Haar measure $dU_C$ in terms of eigenvalues.

point exactly from the equation

$$\frac{\partial}{\partial \theta_j^{(C)}} \Big( S[\sigma_1, \sigma_C|A_1] + S[\sigma_C, \sigma_2|A_2] \Big)$$

$$= \frac{\partial}{\partial \theta} \frac{\delta}{\delta \sigma_C(\theta)} \Big( S[\sigma_1, \sigma_C|A_1] + S[\sigma_C, \sigma_2|A_2] \Big) \Big|_{\theta=\theta_j} = 0. \tag{2.21}$$

Since $S[\sigma_1, \sigma_C|A_1]$ is the action along the classical trajectory connecting $\sigma_1$ and $\sigma_2$, the functional derivative of $S$ equals the corresponding canonical momentum,

$$\frac{\partial}{\partial \theta} \frac{\delta S[\sigma_1, \sigma_C|A_1]}{\delta \sigma_C(\theta)} = v_1(\theta),$$

where $v_1(\theta)$ is the velocity at the end of the trajectory, when $\sigma(t, \theta) = \sigma_C(\theta)$. Similarly,

$$\frac{\partial}{\partial \theta} \frac{\delta S[\sigma_C, \sigma_2|A_2]}{\delta \sigma_C(\theta)} = -v_2(\theta),$$

with $v_2(\theta)$ being the velocity at the beginning of the trajectory connecting $\sigma_C(\theta)$ to $\sigma_2(\theta)$.

Hence, the saddle point condition (2.21) states that $v_1(\theta) = v_2(\theta)$. That is to say, the two trajectories $\sigma_1 \to \sigma_C$ and $\sigma_C \to \sigma_2$ can be joined so that the velocity is continuous. The resulting compound trajectory $\sigma_*(t, \theta)$ is just the solution of the collective field equations (2.17) which describes the original cylinder of area $A$ with the boundary conditions $\sigma_1$ and $\sigma_2$. To find this trajectory, we must solve (2.17) together with the boundary conditions (2.16). Then the saddle point density $\sigma_C(\theta)$ can be determined as $\sigma_*(t = A_1, \theta)$.

Using these arguments we can immediately evaluate any Wilson loop average on a cylinder. In general, a Wilson loop with the winding number $n$ around the contour $C$ equals

$$W_n(C) \equiv \Big\langle \frac{1}{N} \mathrm{Tr}\, U_C^n \Big\rangle$$

$$= \frac{1}{\mathcal{Z}_N(U_{C_1}, U_{C_2}|A)} \int dU_C\, \mathcal{Z}_N(U_{C_1}, U_C|A_1) \Big[\frac{1}{N} \mathrm{Tr}\, U_C^n\Big] \mathcal{Z}_N(U_C, U_{C_2}|A_2). \tag{2.22}$$

At large $N$ the integral in (2.22) is again dominated by a saddle point. Moreover, this is exactly the same saddle point which dominates the integral (2.20). Indeed, the new term in the integrand, $(1/N)\mathrm{Tr}\, U_C^n = (1/N)\sum_{j=1}^{N} \theta_j^n$, is finite as $N \to \infty$ and does not affect the position of the saddle point, which is determined by the balance of exponentially large, $e^{N^2 S}$, terms. As a result, we can calculate $W_n(C)$ at large $N$ evaluating $(1/N)\sum_{j=1}^{N} \theta_j^n$ at the saddle point, to obtain

$$W_n(C) = \int\limits_0^{2\pi} \sigma_*(t = A_1, \theta)\, e^{in\theta}\, d\theta. \tag{2.23}$$

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We have thus proved that the solutions of the collective field equations have the physical meaning of yielding the dominant eigenvalue distributions which, upon Fourier transform, produce the Wilson loops[7] for large $N$ QCD.

## 2.3. The duality relation for QCD on a cylinder.

Now we are in a position to find explicitly the representation which dominates the partition function. Let us first consider the symmetric case $U_{C_1} = U_{C_2}$. Later it will be obvious that our treatment easily gengeralizes to include all other boundary conditions. We will show that $\rho_Y(y)$ satisfies the equation

$$\pi\rho_Y\big( - \pi\sigma_0(\theta)\big) = \theta \tag{2.24}$$

where $\sigma_0(\theta)$ is the solution of collective equations (2.17) taken at the middle of the cylinder, for $t = A/2$.

To illustrate this relation let us reexamine the example of QCD on a sphere [11]. There $\sigma_1(\theta) = \sigma_2(\theta) = \delta(\theta)$ and the solution of the collective field problem (2.16), (2.17) is given by the self-similar evolution of a semicircular eigenvalue distribution

$$\sigma_*(t,\theta) = \frac{1}{\pi}\sqrt{\mu(t) - \frac{\mu^2(t)\theta^2}{4}} \tag{2.25}$$

where the scale $\mu(t)$ changes in time according to

$$\mu(t) = \frac{A}{t(A - t)}. \tag{2.26}$$

Indeed, if we plug the ansatz (2.25) into (2.17) we find that

$$v(t,\theta) = \alpha(t)\theta, \ \dot{\mu} + 2\alpha\mu = 0, \ \dot{\alpha} + \alpha^2 + \frac{\mu^2}{4} = 0 \tag{2.27}$$

whose solution is given by (2.26). At $t = A/2$ we have $\mu(t) = \mu_0 = 4/A$ and

$$\sigma_0(\theta) = \sigma_*\Big(t = \frac{A}{2},\theta\Big) = \frac{2}{\pi A}\sqrt{A - \theta^2}.$$

Using (2.6) we can check that, indeed, $\pi\rho_Y\big( - \pi\sigma_0(\theta)\big) = \theta$ is satisfied.

In general, the solution of collective field equations cannot be written down in an explicit form. However, these equations themselves can be integrated thus reducing the

---

[7] A different formula for Wilson loops on a sphere has been obtained by Boulatov, Daul and Kazakov [15]. Using the duality relation (1.2) it is possible to prove that it is equivalent to (2.23).

problem to a single implicit equation on $\sigma_*(t, \theta)$. To see this, let us introduce a new unknown function

$$f(t, \theta) = v(t, \theta) + i\pi\rho(t, \theta). \tag{2.28}$$

Then the collective field equations (2.17) can be reduced to a single equation

$$\frac{\partial f}{\partial t} + f\frac{\partial f}{\partial \theta} = 0 \tag{2.29}$$

known as the Hopf (or Burgers) equation. Its solution can be found from the implicit formula[8]

$$f(t, \theta) = f_0\big(\theta - t\, f(t, \theta)\big) \tag{2.30}$$

where the function $f_0(\theta)$ represents the initial data, $f_0(\theta) \equiv f(t = 0, \theta)$.

We can use (2.30) to relate $\sigma_0(\theta)$ to the density at the boundary, $\sigma_1(\theta)$. Indeed, if the boundary densities are the same ($\sigma_1(\theta) = \sigma_2(\theta)$) then, by symmetry, the velocity at $t = A/2$ vanishes:

$$v\left(t = \frac{A}{2}, \theta\right) = 0.$$

Thus we can use

$$f_0(\theta) = f\left(t = \frac{A}{2}, \theta\right) = i\pi\sigma_0(\theta)$$

as an initial condition in (2.30). After the time $\Delta t = A/2$ this $f_0$ should evolve into

$$f(t = A, \theta) = v_1(\theta) + i\pi\sigma_1(\theta),$$

with $\sigma(t = A, \theta) = \sigma_1(\theta)$ and some $v_1(\theta) = v(t = A, \theta)$. Thus we obtain the equation

$$i\pi\sigma_0\left[\theta - \frac{A}{2}\big(v_1(\theta) + i\pi\sigma_1(\theta)\big)\right] = v_1(\theta) + i\pi\sigma_1(\theta) \tag{2.31}$$

constraining $\sigma_0(\theta)$ (and also $v_1(\theta)$) as soon as $\sigma_1(\theta)$ is known. Although it would be very difficult to express $\sigma_0$ through $\sigma_1$ in an explicit form, (2.31) contains just what we need to solve for the dominant Young tableau densiity $\rho_Y(y)$ in (2.4).

To do it we will have to evaluate the functional derivatives of $\Xi[\rho_Y, \sigma_1]$ in (2.4). This can be done if we observe that the $U(N)$ characters (2.2) can be represented as analytic continuations of the Itzykson–Zuber integral [16]

$$I_N(A, B) \equiv \int dU\, \mathrm{e}^{N\mathrm{Tr}(AUBU^\dagger)} = \frac{\det\left\|\mathrm{e}^{Na_k b_j}\right\|}{\Delta(a_k)\Delta(b_j)}. \tag{2.32}$$

---

[8] Since $\theta$ is a coordinate on a circle, $f(t, \theta)$ and $f_0(\theta)$ should be regarded as periodic functions of $\theta$ in this formula .

In this formula $A$ and $B$ are arbitrary hermitian matrices, $a_k$ and $b_j$ are their eigenvalues and $\Delta(a_k) \equiv \prod_{i<j}(x_i - x_j)$ is the Van der Monde determinant. Setting $a_k = l_k$, $b_j = \theta_j$ and analytically continuing $a_k \to ia_k$, we see that

$$\det \left\| e^{Na_k b_j} \right\| \to J\left(e^{i\theta_s}\right) \chi_R(U). \tag{2.33}$$

Therefore, we can use the known expressions for the large $N$ limit of the Itzykson–Zuber integral [17] to find the functional $\Xi$ in (2.3). In particular, if as $N \to \infty$ the distributions of $\{a_k\}$ and $\{b_j\}$ converge to smooth functions $\alpha(a)$ and $\beta(b)$, then asymptotically

$$
\begin{aligned}
I_N(A, B) \simeq \exp N^2 \Bigg\{ & S[\alpha, \beta] + \frac{1}{2} \int \alpha(a)\, a^2\, da + \frac{1}{2} \int \beta(b)\, b^2\, db \\
& - \frac{1}{2} \int \alpha(a)\, \alpha(a') \ln |a - a'|\, da\, da' - \frac{1}{2} \int \beta(b)\, \beta(b') \ln |b - b'|\, db\, db' \Bigg\}
\end{aligned}
\tag{2.34}
$$

with some functional $S$ depending on the two distributions $\alpha$ and $\beta$.

Although there is an explicit expression for $S[\alpha, \beta]$, we need to know only its functional derivative

$$\tilde{U}(a) = \frac{\partial}{\partial a} \frac{\delta}{\delta \alpha(a)} S[\alpha, \beta]. \tag{2.35}$$

To determine it we construct a pair of functions $\{G_+(z), G_-(z)\}$ such that

$$
\begin{cases}
G_+\big(G_-(z)\big) = G_-\big(G_+(z)\big) = z, \\
\operatorname{Im} G_+(a + i0) = \pi\alpha(a), \\
\operatorname{Im} G_-(b + i0) = -\pi\beta(b).
\end{cases}
\tag{2.36}
$$

Then

$$\tilde{U}(a) = \operatorname{Re} G_+(a + i0) - a, \tag{2.37}$$

and, in addition,

$$\tilde{V}(b) \equiv \frac{\partial}{\partial b} \frac{\delta}{\delta \beta(b)} S[\alpha, \beta] = -\operatorname{Re} G_-(b + i0) + b. \tag{2.38}$$

If we want to apply these formulas to the calculation of characters, we must be able to perform an analytic continuation $l_k \to i l_k$. This can be done by introducing an additional parameter $t$ and rescaling $l_k \to t l_k$. We will keep $t$ in our formulas up to the very end of the calculations, where we will set $t = i$.

In this case the density of points $y_k(t) = \frac{l_k(t)}{N} = \frac{l_k}{Nt}$ equals[9]

$$\rho_t(y) = \frac{1}{t} \rho\left(\frac{y}{t}\right). \tag{2.39}$$

---

[9] From now on, we omit the subscript "$Y$" in $\rho_Y(y)$.

Therefore, we must distinguish between the derivatives with respect to $\rho(y)$ and $\rho_t(y)$. The quantity which enters the master field equation (2.4) is the derivative with respect to $\rho(y)$,

$$U(y) = \frac{\partial}{\partial y} \frac{\delta}{\delta\rho(y)} S\big[\rho_t(y), \sigma_1(\theta)\big]. \tag{2.40}$$

On the other hand, the constraints (2.36) will yield not $U(y)$, but

$$U_t(y) = \frac{\partial}{\partial y} \frac{\delta}{\delta\rho_t(y)} S\big[\rho_t(y), \sigma_1(\theta)\big]. \tag{2.41}$$

However, these two derivatives are connected by a simple relation

$$U(y) = t\, U_t(ty). \tag{2.42}$$

Indeed, if

$$H_t(y) = \frac{\delta}{\delta\rho_t(y)} S\big[\rho_t(y), \sigma_1(\theta)\big]$$

then

$$\frac{\delta S\big[\rho_t(y), \sigma_1(\theta)\big]}{\delta\rho(y)} = \frac{\delta}{\delta\rho(y)} \int dz\, \delta\rho_t(z)\, H_t(z)$$
$$= \frac{\delta}{\delta\rho(y)} \int dw\, \delta\rho(w)\, H_t(tw) = H_t(ty)$$

where we made the change of variables $w = z/t$ and utilized (2.39). Now it is easy to see that

$$U(y) = \frac{\partial}{\partial y} H_t(ty) = t\, U_t(ty).$$

Therefore, using (2.3), (2.32), (2.33), (2.34) and taking into account that $\ln J(e^{i\theta_s})$ does not contribute to the derivative with respect to $\rho(y)$, we obtain

$$\frac{\partial}{\partial y} \frac{\delta\Xi\big[\rho(y), \sigma_1(\theta)\big]}{\delta\rho(y)} = \frac{\partial}{\partial y} \frac{\delta}{\delta\rho(y)} \left\{ S\big[\rho_t(y), \sigma_1(\theta)\big] + \frac{1}{2} \int \rho_t(y)\, y^2\, dy \right\}\bigg|_{t=i}$$
$$= \Big[U(y) + t^2 y\Big]_{t=i} = \Big[t\, U_t(ty)\Big]_{t=i} - y. \tag{2.43}$$

Now we are able to produce the solution of (2.4). If $\sigma_1 = \sigma_2$, this equation takes the form[10]

$$\mathrm{Re}\, \frac{\partial}{\partial y} \frac{\delta\Xi\big[\rho(y), \sigma_1(\theta)\big]}{\delta\rho(y)} = \frac{A}{2} y. \tag{2.44}$$

We will prove that if $\pi\rho(y)$ is the inverse function of $\pi\sigma_0(\theta)$ then (2.44) is indeed true.

---

[10] We remember that $y = \tilde{y} - \frac{1}{2}$.

To this effect we exibit a pair of functions $\{G_+(z), G_-(z)\}$ which satisfy the constraint (2.36) for $\alpha(a) = \rho_t(a)$ and $\beta(b) = \sigma_1(b)$:

$$G_+(z) = -\frac{A}{2}z + i\pi\rho_t(z), \tag{2.45a}$$

$$G_-(z) = -v_1(\theta) - i\pi\sigma_1(\theta) \tag{2.45b}$$

where $v_1$ is the velocity appearing in (2.31). This would imply that, according to (2.37),

$$U_t(y) = \operatorname{Re} G_+(y + i0) - y = -\frac{A}{2}y - y$$

and using (2.42) we immediately see that (2.44) is satisfied.

The conditions (2.36) are easy to check. While the second and the third condition in (2.36) are satisfied by construction, the first one follows from (2.31). Indeed, we can rewrite (2.31) as

$$\pi\sigma_0\left[\theta + \frac{A}{2}G_-(\theta)\right] = iG_-(\theta).$$

Using $\pi\rho(-\pi\sigma_0(\theta)) = \theta$ and $\rho_t(y)|_{t=i} = \frac{1}{t}\rho(\frac{y}{t})|_{t=i} = -i\rho(-iy)$, we deduce

$$\theta + \frac{A}{2}G_-(\theta) = \pi\rho(-iG_-(\theta)) = i\pi\rho_t(G_-(\theta)).$$

Thus

$$-\frac{A}{2}G_-(\theta) + i\pi\rho_t(G_-(\theta)) = \theta$$

which, together with (2.45a) proves $G_+(G_-(z)) = z$, completing our construction.

If the two boundary densities $\sigma_1$ and $\sigma_2$ are not identical then the partition function may or may not be dominated by a single representation. These two situations can be distinguished by looking at the collective field trajectory connecting $\sigma_1$ to $\sigma_2$. If at some moment $t_0$ the velocity $v(t_0, \theta)$ vanishes simultaneously for all $\theta \in [0, 2\pi]$, then the dominant representation exists and is determined by[11]

$$\pi\rho(-\pi\sigma(t_0, \theta)) = \theta.$$

On the other hand, if this is not the case the saddle point determined by (2.4) shifts to the complex domain and no single representation dominates $\mathcal{Z}[U_{C_1}, U_{C_2}|A]$.

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[11] Now $t_0$ does not have to equal $A/2$.

*2.4. The Douglas–Kazakov phase transition on a cylinder.*

A phase transition similar to the Douglas–Kazakov phase transition on a sphere occurs in the large $N$ QCD on a cylinder.

Technically, this phase transition takes place when the range of values of the function $\pi\rho_Y(y)$ fills the whole interval $[0, 1]$. Then in the strong coupling phase there are continuous intervals of $y$ where $\rho_Y(y) \equiv 1$, while in the weak coupling phase everywhere $\rho_Y(y) < 1$.

Using the duality relation (2.24) we can prove a more physical criterion which determines the transition point. Indeed, since $\pi\rho_Y(y)$ and $\pi\sigma_0(\theta)$ are inverse functions[12] the range of values of $\pi\rho_Y(y)$ coincides with the domain of definition for $\pi\sigma_0(\theta)$. Therefore at the point of phase transition the domain of $\pi\sigma_0$ ranges from $-\pi$ to $\pi$ thus filling the whole circle. On the contrary, in the weak coupling phase this domain is restricted to some interval inside a circle and there is a gap where $\sigma_0(\theta)$ vanishes. Thus this phase transition is precisely of the form found originally for the unitary matrix model [4].

Let us now look at the whole collective field trajectory $\sigma_*(t, \theta)$. As we have seen from (2.17), $\sigma_*$ evolves as the density of the fluid with negative pressure. Since such fluid tends to collapse the moment of time when its velocity vanishes corresponds to the maximum expansion of the fluid. The density profile at this moment is, by definition, $\sigma_0(\theta)$. Therefore, if the support of $\sigma_0$ has a gap, this gap can only increase during further evolution, and the support of $\sigma_*(t, \theta)$ at any other $t$ will have a gap as well. As a result, in the weak coupling phase the collective density exibits the gap at all times, while in the strong coupling phase there are time intervals when the gap disappears. This provides a criterion for the Douglas–Kazakov phase transition which is more general than the condition $\rho_Y(y) = 1$. Indeed, it is applicable even when no single representation dominates the partition function and no $\rho_Y(y)$ exists.

Returning to the QCD on a sphere, we can now construct the collective field trajectory in the strong coupling phase[13]. We are looking for a solution of the Hopf equation (2.29) with the boundary conditions $\mathrm{Im}\, f(t = 0, \theta) = \mathrm{Im}\, f(t = A, \theta) = \pi\delta(\theta)$. Surprisingly, the solution of this problem is quite complicated. Indeed, by the duality formula (2.24) the value of $f(t = A/2, \theta) = \pi\sigma_0(\theta)$ is the inverse function of $\pi\rho_Y(y)$ which, in turn, is given by the elliptic integral (2.7). Since this solution describes the strong coupling phase the support of $\sigma_0$ has no gap (fig. 4).

---

[12] The functional inversion in this context should be carefully understood since, strictly speaking, the inverse of $\pi\rho_Y(y)$ has several branches which should be chosen in an appropriate way. As a consequence, the domain of $\pi\sigma_0$ is $[-\pi, \pi]$ and not $[0, \pi]$.

[13] The trajectory in the weak coupling phase is given by (2.25).
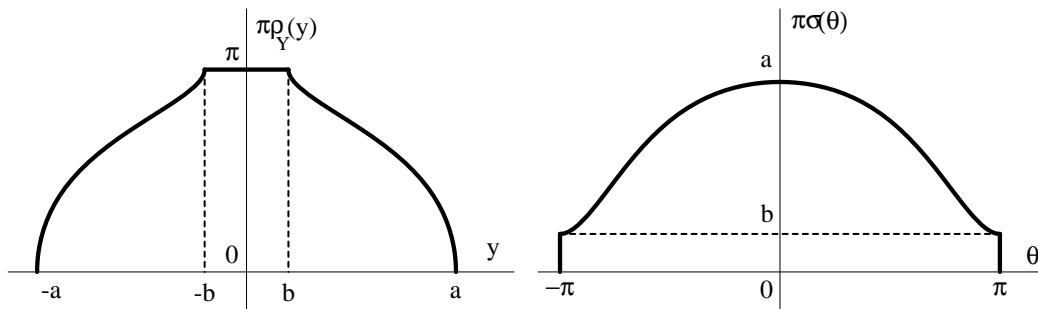
**Fig. 4:** The Young tableau density $\rho_Y(y)$ (left) and the collective field density $\sigma_0(\theta)$ (right) for the QCD on a sphere. In accordance with the duality relation the plots of $\pi\rho_Y(y)$ and $\pi\sigma_0(\theta)$ can be obtained from each other by a reflection with respect to the axis $x = y$.

However, the gap will inevitably appear as $\sigma_0(\theta)$ evolves according to the Hopf equation. After all, the final result of this evolution is the density $\sigma_1(\theta) = \delta(\theta)$ with support consisting of a single point.

Since our boundary conditions are even, so is $\sigma_*(t, \theta)$ for any $t$. Therefore, the gap in the support of $\sigma_*$ will be centered around $\theta = \pi$. In particular, we can find the moment of time when this gap just appears by looking at the solutions of $\sigma_*(t, \theta = \pi) = 0$. We will find that this equation has solutions only if $t \in [0, \frac{1}{2}A - \tau_0(A)] \cup [\frac{1}{2}A + \tau_0(A), A]$. On the other hand, when $t \in ]\frac{1}{2}A - \tau_0(A), \frac{1}{2}A + \tau_0(A)[$ the gap is absent and $\sigma_*(t, \theta) \neq 0$.

Since the Wilson loops are nothing but the Fourier coefficients of $\sigma_*(t, \theta)$, we conclude that the structure of small loops (with area less than the "critical" value $t_{\mathrm{cr}} = \frac{1}{2}A - \tau_0(A)$) resembles the structure of loops in the weak coupling phase, even though we are in the strong coupling phase. On the other hand the loops with larger area are truly characteristic of the strong coupling phase. Physically, we can distinguish between these two types of loops by considering their behavior as the winding number $n$ goes to infinity. Then at fixed area of the contour the strong coupling type of a loop[14] decreases exponentially with $n$

$$W_n(C) = \mathcal{O}(\mathrm{e}^{-\alpha(C)n}), \quad \alpha(C) > 0 \tag{2.46}$$

while the weak coupling type of a loop decreases at most algebraically, as $\mathcal{O}(n^{-p})$. The same is true for the general case of a cylinder.

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[14] By definition, this is the loop for which the corresponding collective field density has no gap. Alternatively, the weak coupling type means that the gap is present. This should not be confused with the strong and weak coupling phases of the Douglas–Kazakov transition.

Continuing our example, let us find how the critical Wilson loop area depends on $A$. Obviously, $\tau_0(A)$ vanishes at the Douglas–Kazakov transition, when $A = \pi^2$, and then grows as $A$ increases.

The evolution of $\sigma_*(t, \theta = \pi)$ is easy to study. We can use (2.30) with the initial condition $f_0(\theta) = i\pi\sigma_0(\theta)$ to obtain[15]

$$i\pi\sigma_0\left[\theta - \left(t - \frac{A}{2}\right)f(t,\theta)\right] = f(t,\theta).$$

However, due to the symmetry $\theta \to -\theta$ the velocity $v(t, \theta)$ vanishes at $\theta = 0$ and $\theta = \pi$. Thus $f(t, \pi) = i\pi\sigma_*(t, \pi)$ and

$$\pi\sigma_0\left[\pi - i\left(t - \frac{A}{2}\right)\pi\sigma_*(t,\pi)\right] = \pi\sigma_*(t,\pi).$$

Finally, using the duality (2.24) we obtain an equation for $z(t) = \pi\sigma_*(t, \pi)$,

$$\pi - i\left(t - \frac{A}{2}\right)z(t) = \pi\rho_Y\big(z(t)\big) \tag{2.47}$$

with $\rho_Y(y)$ given by (2.7).

The solution $z(t)$ we are looking for is a real function of $t$. Moreover, at all times $0 \leq z(t) < b$. Indeed, at $t = t_0 = A/2$ the expansion of our hypothethic fluid is maximal and so is its density at $\theta = \pi$. When the fluid collapses towards $\theta = 0$, $\sigma(t, \pi)$ decreases to zero.

Mathematically, if $z \in [0, b]$

$$\pi\rho_Y(z) = i\frac{2}{az}\sqrt{(a^2 - z^2)(b^2 - z^2)}\,\Pi_1\left(-\frac{b^2}{z^2}, k\right). \tag{2.48}$$

Since in this case $x = -\frac{b^2}{z^2} < -1$, the elliptic integral $\Pi_1(x, k)$ develops an imaginary part. This happens because the poles of the integrand at $t^2 = -\frac{1}{x}$ are now located within the integration region. Using the rule $\frac{1}{x} = \mathcal{P}\frac{1}{x} - i\pi\delta(x)$, we get

$$\Pi_1(x, k) = \frac{1}{2}\int_{-1}^{+1}\frac{du}{1 + xu^2}\frac{1}{\sqrt{(1 - k^2u^2)(1 - u^2)}} - \frac{i\pi}{\sqrt{|x|\left(1 + \frac{k^2}{x}\right)\left(1 + \frac{1}{x}\right)}}. \tag{2.49}$$

Substituting this into (2.48) and using $k = b/a$, we see that

$$\pi\rho_Y(z) = \pi + \frac{i}{az}\sqrt{(a^2 - z^2)(b^2 - z^2)}\int_{-1}^{+1}\frac{du}{(1 + xu^2)\sqrt{(1 - k^2u^2)(1 - u^2)}}.$$

---

[15] Since we set the initial condition at $t_0 = A/2$, the time of evolution in this equation is $t - \frac{1}{2}A$, not $t$.

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Therefore (2.47) reduces to a single real equation

$$\left(t - \frac{A}{2}\right)z = -\frac{1}{az}\sqrt{(a^2 - z^2)(b^2 - z^2)}\int\limits_{-1}^{+1}\frac{du}{(1 - u^2\frac{b^2}{z^2})\sqrt{(1 - k^2u^2)(1 - u^2)}}.\qquad(2.50)$$

It is easy to see that $z = 0$ is always a solution of this equation. However, for certain $t$ it has another solution which can be determined from

$$t - \frac{A}{2} = -\frac{1}{a}\sqrt{(a^2 - z^2)(b^2 - z^2)}\int\limits_{-1}^{+1}\frac{du}{(z^2 - u^2b^2)\sqrt{(1 - k^2u^2)(1 - u^2)}}.$$

Such $z(t)$ decreases as $t - \frac{A}{2}$ deviates from zero, so that when $t$ goes outside of the interval $]\frac{1}{2}A - \tau_0(A), \frac{1}{2}A + \tau_0(A)[$ this solution ceases to exist. Therefore, at $t = \frac{1}{2}A + \tau_0(A)$ we have $z(t) = 0$ and

$$\tau_0(A) = -b\lim_{z\to 0}\int\limits_{-1}^{+1}\frac{du}{(z^2 - u^2b^2)\sqrt{(1 - k^2u^2)(1 - u^2)}}.$$

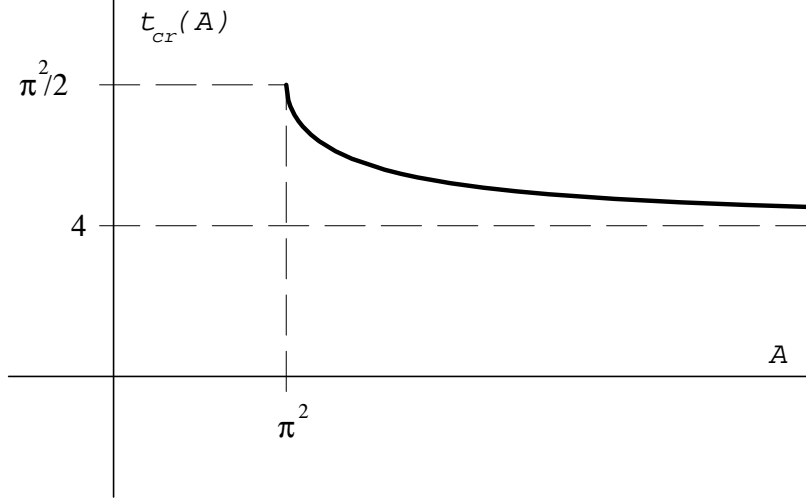Using the $x \to 0$ asymptotics

$$\int\limits_{-1}^{+1}\frac{du}{(u - x)\sqrt{(1 - k^2u^2)(1 - u^2)}} = 2k^2x\int\limits_{0}^{1}\frac{u^2\,du}{\sqrt{(1 - k^2u^2)(1 - u^2)}} + \mathcal{O}(x^2)\qquad(2.51)$$

we conclude that

$$\lim_{z\to 0}\int\limits_{-1}^{+1}\frac{du}{(z^2 - u^2b^2)\sqrt{(1 - k^2u^2)(1 - u^2)}} = -\frac{2k^2}{b}\int\limits_{0}^{1}\frac{u^2\,du}{\sqrt{(1 - k^2u^2)(1 - u^2)}}$$

and finally

$$\tau_0(A) = \frac{1}{ab}\int\limits_{-b}^{b}\frac{\lambda^2\,d\lambda}{\sqrt{(a^2 - \lambda^2)(b^2 - \lambda^2)}} = \frac{2}{b}\left(\mathbf{K}(k) - \mathbf{E}(k)\right).\qquad(2.52)$$

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**Fig. 5:** The critical Wilson loop area $t_{cr}$ as a function of the size of the sphere. As $A \to \infty$ (this corresponds to the QCD on a plane) $t_{cr}$ converges to a finite limit $t_{cr} = 4$.

The dependence of the critical Wilson loop size $t_{cr} = \frac{1}{2}A - \tau_0(A)$ on $A$ is shown in fig. 5. Most remarkably, $t_{cr}$ tends to a finite limit equal to $t_{cr} = 4$ as $A$ goes to infinity. Thus even in the QCD on a plane the eigenvalue distributions for small contours have a gap which disappears as the area enclosed by the contour passes through 4 [18]. That is to say, the master field for QCD on a plane contains some trace of the Douglas–Kazakov phase transition and of the weak coupling phase.

Since the asymptotic property $t_{cr} \to 4$ will be important to us later on, let us prove it. Making the substitution $\lambda = b \cos \varphi$ in (2.52) we get

$$
\tau_0(A) = \frac{2}{a} \int\limits_0^{\pi/2} \frac{\cos^2 \varphi \, d\varphi}{\sqrt{\kappa^2 + \sin^2 \varphi}}
$$

where $\kappa^2 = \frac{a^2}{b^2} - 1 = \frac{k'^2}{k^2}$. At $A \to \infty$, as a consequence of (2.9), $k'^2 \simeq 16e^{-A/4} \to 0$ and thus $\kappa^2 = 16e^{-A/4} + \ldots$. In this limit we can break up our integral into two parts and construct the asymptotic expansions for both of them:

$$
\int\limits_0^{\pi/2} \frac{\cos^2 \varphi \, d\varphi}{\sqrt{\kappa^2 + \sin^2 \varphi}} = \int\limits_0^{\epsilon} \frac{d\varphi}{\sqrt{\kappa^2 + \varphi^2}} + \int\limits_{\epsilon}^{\pi/2} \frac{\cos^2 \varphi}{\sin \varphi} \, d\varphi
$$

$$
= \operatorname{arsinh}\frac{\epsilon}{\kappa} + \ln \cotan \frac{\epsilon}{2} - 1 + \mathcal{O}(\epsilon) + \mathcal{O}(\kappa^2)
$$

$$
= -1 + \ln \frac{4}{\kappa} + \ldots = -1 + \frac{A}{8} + \mathcal{O}(e^{-A/4})
$$

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In these formulas we chose an auxiliary small quantity $\epsilon$ so that $\kappa \ll \epsilon \ll 1$. Then at the end $\epsilon$ drops out of the final result, leaving us with the desired asymptotics.

Now from (2.9) $a = \frac{1}{2} + \mathcal{O}(e^{-A/4})$ and thus finally

$$\tau_0(A) = \frac{A}{2} - 4 + \mathcal{O}(e^{-A/4})$$

leading to

$$t_{\mathrm{cr}}(A) = 4 + \mathcal{O}(e^{-A/4}) \tag{2.53}$$

as claimed.

Let us now demonstrate that the asymptotic behaviour of Wilson loops for large winding numbers $n$ is related to the Douglas–Kazakov transition. The most convenient way to do this is to continue considering the planar case. For one, as we saw, the QCD on a plane retains some trace of the weak coupling phase (the gap in the eigenvalue distribution for loops with area less than 4). On the other hand, all Wilson loops on a plane can be evaluated explicitly, making the analysis very simple.

Indeed, the Wilson loops on a plane equal [8][19]

$$W_n(A) = \left\langle \frac{1}{N} \mathrm{Tr}\, U_C^n \right\rangle = \frac{1}{n} e^{-\frac{nA}{2}} L_{n-1}^{(1)}(An) \tag{2.54}$$

where $L_{n-1}^{(1)}(x)$ are the so-called associated Laguerre polynomials

$$L_{n-1}^{(1)}(x) = \sum_{k=0}^{n-1} (-x)^k \frac{n!}{(n-k-1)!\, k!\, (k+1)!} = \oint \frac{dt}{2\pi i} \left(1 + \frac{1}{t}\right)^n e^{-xt}. \tag{2.55}$$

Using this expression we can investigate the behaviour of $W_n(A)$ as $n \to \infty$ at fixed $A$. We will see that $W_n(A)$ decays exponentially in $n$ if $A > 4$, but as a power of $n$ if $A < 4$. To prove this we need to know the asymptotic behaviour of the Laguerre poynomials as $x = An \to \infty$. This can be found by treating the integral in (2.55) using the saddle point method[16]. The asymptotics of this integral is governed by the minimum of the functional

$$\Phi(t) = \frac{x}{n} t - \ln\left(1 + \frac{1}{t}\right). \tag{2.56}$$

This minimum is reached at

$$t_\pm = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{n}{x}} = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{1}{A}}.$$

---

[16] This calculation is remarkably similar to the calculation of instanton contributions in the weak coupling phase which are expressed as Laguerre polynomials as well [3]. This is not suprising since these are different manifestations of the same phase transition.

If $A > 4$ then both of $t_+$ and $t_-$ are real. Inn this case it is possible to show that the dominant contribution comes from the region around $t = t_+$ and equals

$$L_{n-1}^{(1)}(x) \simeq \frac{1}{\sqrt{2\pi n |\Phi''(t_+)|}} e^{-n\Phi(t_+)} = \frac{A}{\sqrt{2\pi n}} \left(1 - \frac{4}{A}\right)^{-\frac{1}{4}} e^{\frac{nA}{2}[1-\gamma(4/A)]} \tag{2.57}$$

where we have introduced a special function

$$\gamma(x) = \sqrt{1-x} - \frac{x}{2} \ln \frac{1+\sqrt{1-x}}{1-\sqrt{1-x}} = 2\sqrt{1-x} \sum_{s=1}^{\infty} \frac{(1-x)^{2s}}{4s^2 - 1} > 0. \tag{2.58}$$

Thus at $A > 4$

$$W_n(A) \simeq (-)^{n-1} \frac{A}{\sqrt{2\pi n^3}} \left(1 - \frac{4}{A}\right)^{-\frac{1}{4}} e^{-\frac{nA}{2}\gamma\left(\frac{4}{A}\right)} \tag{2.59}$$

so that as $n \to \infty$, $W_n(A)$ decays exponentially with the index $\alpha(A) = \frac{A}{2}\gamma(4/A) > 0$.

On the contrary, if $A < 4$ the saddle points $t_+$ and $t_-$ are complex conjugate and make comparable contributions. Then

$$L_{n-1}^{(1)}(x) \simeq (-)^{n-1} \frac{1}{\sqrt{2\pi n |\Phi''(t_+)|}} 2\mathrm{Re}\left\{e^{-n\Phi(t_+)-i\pi/4}\right\}$$

$$= (-)^{n-1} \sqrt{\frac{2}{\pi n}} A \left(\frac{4}{A} - 1\right)^{-\frac{1}{4}} e^{\frac{nA}{2}} \cos\left\{\frac{nA}{2}\Gamma\left(\frac{4}{A}\right) + \frac{\pi}{4}\right\} \tag{2.60}$$

where $\Gamma(x) = \sqrt{x-1} - x\arctan\sqrt{x-1}$. As a result, for $A < 4$ the Wilson loops

$$W_n(A) \simeq \sqrt{\frac{2}{\pi n^3}} A \left(\frac{4}{A} - 1\right)^{-\frac{1}{4}} \cos\left\{\frac{nA}{2}\Gamma\left(\frac{4}{A}\right) + \frac{\pi}{4}\right\} \tag{2.61}$$

do not decay exponentially. Rather, they oscillate with $n$ and the amplitude of oscillation decays as $n^{-3/2}$. Such behaviour is similar to what we encounter in the weak coupling phase on a sphere before the Douglas–Kazakov transition occurs[17].

To summarize, the phase transition in continuum two-dimensional QCD exibits itself through physical observables in several ways. First, the density of eigenvalues for Wilson loop matrices develops a gap in the weak coupling phase. Second, the asymptotic behaviour of Wilson loops with large winding numbers is different in the two phases. Finally, the Young tableau of the dominant representation also can be expressed in terms of physical observables using the duality formula (2.24).

---

[17] Let us emphasize that the absence of a gap and the exponential decay of Wilson loops are not automatic consequences of each other. In fact, the $n \to \infty$ behaviour of $W_n(A)$ probes the degree of smoothness of the eigenvalue distribution. Since when there is no gap the distribution is infinitely smooth the Wilson loops decay faster than any power of $n$. On the other hand, if we do have a gap, then at the edge of the support of the eigenvalues higher derivatives become discontinuous, and the loops decay powerlike.

## 3. Instanton contributions to the Wilson loops.

As we saw in the previous section, the Douglas–Kazakov phase transition is associated with the development of a gap in the eigenvalue distributions for Wilson matrices. In this section we will argue that this phenomenon, very much like the original Douglas–Kazakov transition on a sphere at $A_{\rm cr} = \pi^2$, is induced by instantons.

Indeed, it is known that the partition function of $QCD_2$ can be repesented exactly as a sum of contributions localized at the instantons, that is, the classical solutions of the theory [20][18]. As it turns out, in the weak coupling phase, at $A_{\rm cr} < \pi^2$, the dominant term comes from the expansion around the perturbative vacuum, $\mathcal{A}_\mu(x) = 0$, while in the strong coupling phase, $A_{\rm cr} > \pi^2$, a certain nontrivial instanton configuration will dominate.

A similar analysis can be carried out for Wilson loops. We can anticipate that the average of a Wilson loop will be given just by its classical value as determined by the dominant instanton configuration[19] modified by quantum corrections. The easiest way to determine these is to perform a Poisson resummation in the original formula for Wilson loops in terms of the representations of the gauge group. On the sphere, for a simple contour which divides it into two patches of areas $A_1$ and $A_2$, the Wilson loop is given by

$$W_n(A_1, A_2) \equiv \left\langle \frac{1}{N} {\rm tr} U^n \right\rangle = \frac{1}{\mathcal{Z}(A_1 + A_2)} \sum_{R,S} d_R d_S$$

$$\times \left[ \frac{1}{N} \int dU \, ({\rm tr}\, U^n) \, \chi_R(U) \chi_S(U^\dagger) \right] e^{-\frac{A_1}{2N} C_2(l^{(R)}) - \frac{A_2}{2N} C_2(l^{(S)})}. \tag{3.1}$$

In this formula $R$ and $S$ label all irreducible representations of the gauge group, $d_R$ and $d_S$ are their dimensions and $\chi(U)$ are the characters given by (2.2). (3.1) can be derived easily by gluing two discs of areas $A_1$ and $A_2$ along the loop.

Labelling $R$ and $S$ by two sets of integers, $l_1^{(R)} > l_2^{(R)} > \ldots > l_N^{(R)}$ and $l_1^{(S)} > l_2^{(S)} > \ldots > l_N^{(S)}$ and transforming $\int dU$ to an integral over the eigenvalues of $U$ we get

$$W_n(A_1, A_2) \mathcal{Z}(A_1 + A_2) = \sum_{\substack{l_1^{(R)} > l_2^{(R)} > \ldots > l_N^{(R)} \\ l_1^{(S)} > l_2^{(S)} > \ldots > l_N^{(S)}}} \Delta(l_i^{(R)}) \Delta(l_j^{(S)}) e^{-\frac{A_1}{2N} C_2(l^{(R)}) - \frac{A_2}{2N} C_2(l^{(S)})}$$

$$\times \int d\theta_1 \ldots d\theta_N \left( \frac{1}{N} \sum_{k=1}^N e^{in\theta_k} \right) \det \left\| e^{il_i^{(R)} \theta_p} \right\| \det \left\| e^{-il_j^{(S)} \theta_q} \right\|, \tag{3.2}$$

---

[18] By instantons we mean all possible solutions of the classical Yang–Mills equations even if they are unstable.

[19] This is the value we obtain if we substitute the classical $\mathcal{A}_\mu(x)$ for the dominant instanton into ${\rm tr}\, U = {\rm tr}\, \mathcal{P} \exp \oint \mathcal{A}_\mu(x) \, dx^\mu$.

with $\Delta(x_i) \equiv \prod_{i<j}(x_i - x_j)$. Now we can expand the determinants and use the antisymmetry of $\Delta(l)$ with respect to the permutation of $l_i$'s to remove the ordering restriction on the $l$'s. We obtain

$$W_n(A_1, A_2)\mathcal{Z}(A_1 + A_2) = \frac{1}{N}\sum_{k=1}^{N}\sum_{\substack{l_i^{(R)}, l_j^{(S)} \\ unrestricted}} \Delta(l_i^{(R)})\Delta(l_j^{(S)})$$

$$\times e^{-\frac{A_1}{2N}C_2(l^{(R)}) - \frac{A_2}{2N}C_2(l^{(S)})} \int d\theta_1\, d\theta_2 \ldots d\theta_N\, e^{in\theta_k} \prod_{p=1}^{N} e^{il_p^{(R)}\theta_p} \prod_{q=1}^{N} e^{-il_q^{(S)}\theta_q} \qquad (3.3)$$

$$= \frac{1}{N}\sum_{k=1}^{N}\sum_{l_i^{(R)}} \Delta(l_i^{(R)})\Delta(l_j^{(S)})\, e^{-\frac{A_1}{2N}C_2(l^{(R)}) - \frac{A_2}{2N}C_2(l^{(S)})}$$

where now the summation is only over $l_i^{(R)}$, the $l_j^{(S)}$ being determined by $l_i^{(S)} = l_j^{(R)} + n\delta_{j,k}$. That is to say, $l_i^{(S)} = l_j^{(R)}$ for all $i$ except for $i = k$, when $l_k^{(S)} = l_k^{(R)} + n$.

To represent (3.3) as a sum of instanton contributions let us transform it using the Poisson resummation formula,

$$\sum_{n_1,\ldots,n_N=-\infty}^{+\infty} f(n_1,\ldots,n_N) = \sum_{m_1,\ldots,m_N=-\infty}^{+\infty} F(2\pi m_1,\ldots,2\pi m_N) \qquad (3.4)$$

where

$$F(p_1,\ldots,p_N) = \int_{-\infty}^{+\infty} f(x_1,\ldots,x_N)\, e^{i(p_1 x_1 + \ldots + p_N x_N)} dx_1 \ldots dx_N.$$

First we recall that, up to an irrelevant uniform shift of $l_k$'s by $(N-1)/2$,

$$C_2(l) = \sum_{i=1}^{N} l_i^2.$$

Thus

$$A_1 C_2(l^{(R)}) + A_2 C_2(l^{(S)})$$

$$= \frac{A_1 + A_2}{2}\left[C_2(l^{(R)}) + C_2(l^{(S)})\right] + \frac{A_1 - A_2}{2}\left[C_2(l^{(R)}) - C_2(l^{(S)})\right]$$

$$= \frac{A_1 + A_2}{2}\left[C_2(l^{(R)}) + C_2(l^{(S)})\right] + \frac{A_2 - A_1}{2}(n^2 + 2nl_k^{(R)}).$$

and we can represent our Wilson loops in the form

$$W_n(A_1, A_2)\mathcal{Z}(A_1 + A_2)$$

$$= e^{-\frac{n^2}{4N}(A_2 - A_1)}\frac{1}{N}\sum_{k=1}^{N}\sum_{l_i^{(R)}} \varphi(\{l_i^{(R)}\})\varphi(\{l_i^{(R)} + \delta_{i,k}n\})\, e^{-\frac{n}{2N}(A_2 - A_1)l_k^{(R)}},$$

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where we have introduced a new function of $N$ variables $\{l_j\}$

$$\varphi(\{l_j\}) = \Delta(l_j)\, e^{-\frac{A}{4N}\sum_{p=1}^{N} l_p^2}, \tag{3.5}$$

$A = A_1 + A_2$ being the total area of the sphere.

To apply the Poisson formula (3.4) we have to find the Fourier transform of the function under summation:

$$F(p_1,\ldots,p_N)$$

$$= \int_{-\infty}^{+\infty} dx_1 \ldots dx_N\, \varphi(\{x_i\})\, \varphi(\{x_i + \delta_{i,k} n\})\, e^{-\frac{n}{2N}(A_2 - A_1)x_k}\, e^{i(p_1 x_1 + \ldots + p_N x_N)}$$

$$= \int_{-\infty}^{+\infty} dx_1 \ldots dx_N\, \varphi(\{x_i\})\, \varphi(\{x_i + \delta_{i,k} n\})\, e^{i(\tilde{p}_1 x_1 + \ldots + \tilde{p}_N x_N)}$$

where $\tilde{p}_j = p_j + (in/2N)(A_2 - A_1)\delta_{i,k}$. Recalling that the Fourier transform of a product is a convolution of the individual Fourier transforms we get

$$F(p_1,\ldots,p_N) = \int_{-\infty}^{+\infty} dy_1 \ldots dy_N\, \psi\left(\left\{\frac{\tilde{p}_j + y_j}{2}\right\}\right) \psi\left(\left\{\frac{\tilde{p}_j - y_j}{2}\right\}\right) e^{-i(\tilde{p}_k - y_k)n/2}, \tag{3.6}$$

where $\psi(\{p\})$ is the Fourier transform of $\varphi(\{x\})$,

$$\psi(\{p\}) \equiv \int_{-\infty}^{+\infty} dx_1 \ldots dx_N\, \varphi(\{x_i\})\, e^{i(p_1 x_1 + \ldots + p_N x_N)} = C_N \Delta(p_i)\, e^{-\frac{N}{A}\sum_{i=1}^{N} p_i^2} \tag{3.7}$$

with some $N$-dependent constant which will cancel later on[20]. The phase factor in (3.6) is due to the shift of $x_k \to x_k + n$ in $\varphi(\{x_i + \delta_{i,k} n\})$. Thus

$$F(p_1,\ldots,p_N) = \tilde{C}_N\, e^{-in\tilde{p}_k/2}\, e^{-\frac{N}{2A}\sum_{j=1}^{N} p_j^2}$$

$$\int_{-\infty}^{+\infty} dy_1 \ldots dy_N \left[\prod_{i<j}(\tilde{p}_{ij}^2 - y_{ij}^2)\right] e^{iny_k/2}\, e^{-\frac{N}{2A}\sum_{j=1}^{N} y_j^2}$$

---

[20] To derive this one represents $\Delta(l_j)$ in (3.5) as a Van der Monde determinant and performs the integrations explicitly. The answer can be simplified to give (3.7) if we remember that one can add rows of a determinant without changing it. See [5] for the details.

where $\tilde{p}_{ij} \equiv \tilde{p}_i - \tilde{p}_j$, $y_{ij} \equiv y_i - y_j$ and $\tilde{C}_N$ is another constant, $\tilde{C}_N = C_N^2/2^{N(N-1)}$.

Combining the pieces we finally obtain the following representation for the Wilson loops:

$$
\begin{aligned}
&W_n(A_1, A_2)\mathcal{Z}(A_1 + A_2) \\
&= \frac{1}{N} \sum_{k=1}^{N} \sum_{m_1,\ldots,m_N=-\infty}^{+\infty} F\left(2\pi m_j + \delta_{j,k}\frac{in}{2N}(A_2 - A_1)\right) \\
&= \frac{1}{N} \sum_{k=1}^{N} \sum_{m_1,\ldots,m_N=-\infty}^{+\infty} \mathrm{e}^{-\frac{2\pi^2 N}{A}\sum_{j=1}^{N} m_j^2} \, \mathrm{e}^{-2\pi i n m_k A_2/A} \, \mathrm{e}^{n^2(A_1-A_2)^2/8AN} \qquad (3.8) \\
&\quad \times \left\{ \int_{-\infty}^{+\infty} dy_1 \ldots dy_N \, \mathrm{e}^{iny_k/2} \prod_{i<j}[4\pi^2\tilde{m}_{ij}^2 - y_{ij}^2] \, \mathrm{e}^{-\frac{N}{2A}\sum_{j=1}^{N} y_j^2} \right\}
\end{aligned}
$$

where $\tilde{m}_{ij} \equiv \tilde{m}_i - \tilde{m}_j$ and $\tilde{m}_j = m_j + in\delta_{j,k}(A_2 - A_1)/4\pi N$.

This formula has an interpretation in terms of instantons. Indeed, the instantons of $\mathrm{QCD}_2$ can be labelled by $N$ integers $m_1, \ldots, m_N$ and have the action

$$
S_{\mathrm{inst}}(m_1, \ldots, m_N) = \frac{2\pi^2 N}{A} \sum_{j=1}^{N} m_j^2. \qquad (3.9)
$$

The corresponding field configuration is just a collection of Dirac monopoles

$$
A_\mu(x) = \begin{pmatrix} m_1\mathcal{A}_\mu^0(x) & 0 & \ldots & 0 \\ 0 & m_2\mathcal{A}_\mu^0(x) & \ldots & 0 \\ 0 & \ldots & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & m_N\mathcal{A}_\mu^0(x) \end{pmatrix} \qquad (3.10)
$$

where $\mathcal{A}_\mu^0(x) = \mathcal{A}_\mu^0(\Theta, \phi)$ is the Dirac monopole potential,

$$
\mathcal{A}_\Theta^0(\Theta, \phi) = 0, \quad \mathcal{A}_\phi^0(\Theta, \phi) = \frac{1 - \cos\Theta}{2},
$$

$\Theta$ and $\phi$ being the polar (spherical) coordinates on $S^2$. We can see that the terms $\mathrm{e}^{2\pi i n m_k A_2/A}$ in (3.8) represent the classical contribution of the field configurations (3.10) to the Wilson loop while the term

$$
\zeta_n^{(k)}(m_1, \ldots, m_N) = \int_{-\infty}^{+\infty} dy_1 \ldots dy_N \, \mathrm{e}^{iny_k/2} \prod_{i<j}[4\pi^2\tilde{m}_{ij}^2 - y_{ij}^2] \, \mathrm{e}^{-\frac{N}{2A}\sum_{j=1}^{N} y_j^2} \qquad (3.11)
$$

represents quantum corrections due to the fluctuations around the instanton. Thus our final result can be represented in the form

$$W_n(A_1, A_2) = \frac{1}{\mathcal{Z}(A)} \sum_{\substack{all \\ instantons}} e^{-S_{inst}} \left[ \frac{1}{N} \sum_{k=1}^{N} e^{-2\pi i n m_k A_2/A} \zeta_n^{(k)} \right] e^{n^2(A_1-A_2)^2/8AN}. \quad (3.12)$$

In fact, if we consider (3.8) with $n = 0$, we will reproduce the instanton representation for $\mathcal{Z}(A)$ constructed by Minahan and Polychronakos [5],

$$\mathcal{Z} = \sum_{m_1,\ldots,m_N=-\infty}^{+\infty} w(m_1, \ldots, m_N) e^{-S(m_1,\ldots,m_N)} \quad (3.13)$$

where

$$w(m_1, \ldots, m_N) = \int_{-\infty}^{+\infty} \prod_{i=1}^{N} \prod_{i<j=1}^{N} \left( 4\pi^2 m_{ij}^2 - y_{ij}^2 \right) e^{-\frac{N}{2A} \sum_{i=1}^{N} y_i^2} \quad (3.14)$$

is the statistical weight of the configuration $(m_1, \ldots, m_N)$.

In the large $N$ limit we expect that the sum in (3.13) is dominated by a certain configuration $(m_1^{(0)}, \ldots, m_N^{(0)})$ depending on the area $A$. (For example, in the weak coupling phase the dominant configuration is $m_1^{(0)} = \ldots = m_N^{(0)} = 0$.) As $N \to \infty$ for any such configuration the integrals (3.11) and (3.14) are dominated by some saddle points in $y_i$'s. Moreover, the saddle points for these two integrals are exactly the same, as they are determined by the balance of $e^{\mathcal{O}(N^2)}$ terms which are common for both integrals. Thus, in the large $N$ limit, we can determine $\zeta_n^{(k)}/w$ by substituting the values $y_i = y_i^{(0)}$ into the ratio of the integrands:

$$\frac{\zeta_n^{(k)}}{w} = e^{iny_k^{(0)}/2} \prod_{i<j} \left( 1 - \frac{4\pi^2(m_{ij}^2 - \tilde{m}_{ij}^2)}{4\pi^2 m_{ij}^2 - y_{ij}^{(0)2}} \right).$$

Note that, if $A_1 = A_2 = A/2$ (that is, if we are considering the Wilson loop around the equator) then $m_i = \tilde{m}_i$ and

$$\frac{\zeta_n^{(k)}}{w} = e^{iny_k^{(0)}/2}.$$

Therefore in the leading large $N$ order

$$W_n(A_1, A_2) = \frac{1}{N} \sum_{k=1}^{N} e^{i\pi n m_k} \frac{\zeta_n^{(k)}}{w} = \frac{1}{N} \sum_{k=1}^{N} e^{in(\pi m_k + y_k^{(0)}/2)}.$$

Thus we see that the large $N$ density $r(\eta)$ describing the distribution of numbers $\eta_k = y_k^{(0)}/2 + \pi m_k$ is nothing but the eigenvalue distribution for the equatorial Wilson loop, $\sigma_0(\theta)$. In the weak coupling phase, where all $m_k = 0$, this identity is easy to check directly.

Let us also mention that the distribution of $(m_1^{(0)}, \ldots, m_N^{(0)})$ for the dominant instanton configuration in the strong coupling phase can be determined from the formula

$$\sum_{m=-\infty}^{+\infty} p_m e^{-imq} = \int_{-b}^{+b} dh \, \exp\left\{ -\frac{q}{\pi}\left[ \fint \frac{\rho_Y(y)\,dy}{h-y} - Ah \right] \right\} = 1 - 2b + \int_{-b}^{+b} dh \, e^{iq\rho_Y(h)} \quad (3.15)$$

where $q \in [-\pi, \pi]$, $p_m$ is the probability to find an integer $m$ among the set $\{m_1^{(0)}, \ldots, m_N^{(0)}\}$ and $\rho_Y(y)$ is the Young tableau density given by (2.7). This can be derived using the Poisson resummation in a way similar to (3.8).

As we can see from (3.12), the quantum corrections make important quantitative contributions to Wilson loops. However, the physics of the phase transition can be understood even if we neglect them. Consider the classical contribution to the Wilson loop,

$$W^{\rm cl}(A_1, A_2) = \frac{1}{N} \sum_{k=1}^{N} e^{-2\pi i n m_k A_2/A}. \quad (3.16)$$

The effective eigenvalue distribution for such a loop will have a gap if the angles $\theta_k = 2\pi m_k A_2/A$ are restricted to a certain domain inside of a circle, $|\theta_k| \le \theta_{\max} < \pi$ for all $k$. Since $\theta_{\max} = 2\pi m_{\max} A_2/A$, we get the condition on the area of our loop

$$A_2 < \frac{A}{2m_{\max}}.$$

As $A_2$ increases this inequality may be violated and the eigenvalue gap will disappear[21]. The exact value of $A_2$ when this will occur will be smaller than $A/2m_{\max}$ since quantum fluctuations cause some additional widening of the eigenvalue distribution.

In general, for $A_2 = 0$ we always have $\theta_k = 0$, even in four-dimensional QCD, since the corresponding Wilson matrix is the identity. On the other hand, as the area of the loop becomes very large the gauge fields at the distant points of the contour are uncorrelated and the distribution of angles becomes uniform: $\sigma(\theta) = 1/2\pi$. This is also true in the four-dimensional case. We can see that while zero-area loops have an eigenvalue gap, the infinitely large ones do not. This might be a piece of evidence for a large $N$ phase transition in QCD in *any* dimension. However, we also saw that the quantum corrections tend to broaden the eigenvalue distribution. In fact, it might happen that for any small nonzero area the distributions develop infinitesimal tails and do not have a gap anymore. The

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[21] On the other hand, in the weak coupling phase all $m_k = 0$ and the inequality $A_2 m_{\max} < A/2$ is always satisfied, consistent with the fact that there is always a gap.

nontrivial fact about $\text{QCD}_2$ is that such a phenomenon does not occur there. The reason is that the Hopf equation, governing the evolution of the eigenvalue densities, essentially prohibits the development of such smooth tails. Rather, its generic solution has a (moving) edge and vanishes as $\sigma(\theta, t) \sim \sqrt{\theta - \theta_{cr}(t)}$ near it. Whether or not the situation is similar in higher dimensions is unclear.

One could contemplate analysing the 4d Wilson loops for small area by perturbation theory, since in this case the asymptotic freedom applies. However, to obtain the eigenvalue distribution one would have to evaluate the Wilson loops with arbitrarily high winding numbers $n$. This corresponds to multiplying the charge flowing around the loop by $n$, and for high $n$ this takes us out of the domain where we can rely on perturbation theory.

Let us emphasize that the phase transition in the eigenvalue densities of the Wilson loops does not yet imply that there should be a singularity in the free energy of the theory. For example, this is the case with QCD on a two-dimensional plane. But such a phenomenon might imply a phase transition for the free energy of QCD in a box of a finite size[22] (say, for a sphere in the case of $\text{QCD}_2$). In general, such transitions would mean that the string representation of $\text{QCD}_4$, if constructed, will have only a certain domain of validity. On the other hand, it may offer an explanation how the short distance behavoiur of QCD which is described in terms of particles, agrees with the string description.

## 4. Exact critical area for the phase transition on a disc.

As another application of the duality formula (2.24) we will calculate the critical area for QCD on a disc. A disc can be regarded as a particular case of a cylinder with one of the boundary holonomy matrices set to identity, $U_{C_2} = I$, that is to say $\sigma_2(\theta) = \delta(\theta)$.

Quite remarkably, in this case the boundary value problem for the Euler equations (2.16), (2.17) can be reduced to a Cauchy problem and then solved explicitly using (2.30). To this end we need to know how to specify the velocity $v(t = 0, \theta) \equiv v_1(\theta)$ at the beginning of the trajectory so that at the end, when $t = A$, the second boundary condition (2.16) is satisfied, $\sigma(\theta)|_{t=A} = \sigma_2(\theta) = \delta(\theta)$.

While such a problem cannot be solved explicitly for an arbitrary $\sigma_2(\theta)$, in the particular case $\sigma_2(\theta) = \delta(\theta)$ the necessary velocity is given by

$$v_1(\theta) = -\frac{\theta}{A} + \int \frac{\sigma_1(\theta') \, d\theta'}{\theta - \theta'}. \tag{4.1}$$

---

[22] See also the recent paper by Douglas [21] who draws similar conclusions.

To see why this is true let us recall that the boundary value problem identical to (2.16), (2.17) occurs in the calculation of the Itzykson–Zuber integral. The correspondence between these two problems is established by $\sigma_1(\theta) \leftrightarrow \alpha(\theta)$, $\sigma_2(\theta) \leftrightarrow \beta(\theta)$ (see (2.36) and the analysis in the Appendix B). In the case $\sigma_2(\theta) = \delta(\theta)$ we would have $\beta(b) = \delta(b)$ which means that the argument $B$ of the Itzykson–Zuber integral is simply zero[23]. Then the integral itself is obviously equal to one, $I_N(A, B = 0) = 1$. On the other hand, we can apply the asymptotic formula (2.34) to obtain

$$S\big[\alpha(a), \beta(b) \equiv \delta(b)\big] = -\frac{1}{2} \int \alpha(a)\, a^2\, da + \frac{1}{2} \int \alpha(a)\, \alpha(a') \ln|a - a'|\, da\, da'. \qquad (4.2)$$

The functional $S$ is the action along the classical trajectory connecting $\alpha(a)$ and $\beta(b)$ during the unit time interval, $t = 1$. The velocity at the beginning of the trajectory can be found as

$$v(a) = \frac{\partial}{\partial a} \frac{\delta S[\alpha, \beta]}{\delta \alpha(a)} = -a + \int \frac{\alpha(a')\, da'}{a - a'}.$$

If the time interval is not equal to $t = 1$, this formula can be easily generalized[24]

$$v(a) = -\frac{a}{t} + \int \frac{\alpha(a')\, da'}{a - a'}$$

finally proving (4.1).

Since the Euler equations (2.17) are equivalent to the Hopf equation (2.29), we can now set up the Cauchy problem for (2.29) with the initial condition

$$f(t = 0, \theta) \equiv f_0(\theta) = -\frac{\theta}{A} + \int \frac{\sigma_1(\theta')\, d\theta'}{\theta - \theta'} + i\pi\sigma_1(\theta) = -\frac{\theta}{A} + \int \frac{\sigma_1(\theta')\, d\theta'}{\theta - \theta'}. \qquad (4.3)$$

Then the solution is determined by (2.30) which for this particular case translates into

$$\left(1 - \frac{t}{A}\right) f(t, \theta) = -\frac{\theta}{A} + \int \frac{\sigma_1(\theta')\, d\theta'}{\theta - \theta' - t f(t, \theta)}. \qquad (4.4)$$

This equation can be used to determine $f(t, \theta)$ and, consequently, $\sigma(t, \theta)$, for any specific $\sigma_1$. However, the problem of the Douglas–Kazakov phase transition can be solved without doing so. Indeed, as we proved in the previous subsection, in order to test for the transition

---

[23] Note the important distinction from the matrix $U_{C_2}$, which is the *identity* matrix for $\sigma_2(\theta) = \delta(\theta)$.

[24] To do this one considers the integral $I_N(A, B|t) = \int dU\, e^{\frac{N}{t} \text{Tr}(AUBU^\dagger)}$. Its asymptotics is obtained by replacing $\frac{1}{2} \int \alpha(a)\, a^2\, da \to \frac{1}{2t} \int \alpha(a)\, a^2\, da$ and $\frac{1}{2} \int \beta(b)\, b^2\, db \to \frac{1}{2t} \int \beta(b)\, b^2\, db$ in the formula (2.34).

it is sufficient to find out whether or not the support of $\sigma(t,\theta)$ develops a gap at any time $t \in [0, A]$. This allows one to obtain an explicit formula for the area of the disk when the transition takes place.

To simplify the analysis, let us consider the case of a symmetric $\sigma_1(\theta)$, so that $\sigma_1(\theta) = \sigma_1(-\theta)$. If the support of $\sigma_1$ is the whole circle $S^1 = [-\pi, \pi]$ then in the process of evolution this support must always develop a gap. Indeed, the final result of such evolution is a delta function with support which covers only a single point $\theta = 0$.

Such picture is characteristic of the strong coupling phase. Therefore, in this situation we never have any weak coupling phase and the transition is impossible. A different picture arises if the support of $\sigma_1$ covers only a part of a circle $[-b, b]$ with $b < \pi$. Then the Euler evolution can cause this support to expand with $t$ before it starts contracting to $\theta = 0$. If this expansion stops before the edge of the support, $b(t)$, reaches $\pi$ then the system stays in the weak coupling phase. Otherwise the transition to the strong coupling phase will occur.

At the moment $t_c$ when $b(t)$ reaches its maximum $f(t_c, \theta = b(t_c)) = 0$. Indeed, $\sigma(t_c, \theta)|_{\theta=b(t_c)} = 0$ because $b(t_c)$ is the endpoint of the support of $\sigma$. Moreover, $v(t_c, b(t_c)) = 0$ because exactly at $t = t_c$ the support has just stopped expanding and the velocity at its edge is zero[25]. Thus we may determine $\theta_c = b(t_c)$ using (4.4) with $f = 0$:

$$\frac{\theta_c}{A} = \int \frac{\sigma_1(\theta')\, d\theta'}{\theta_c - \theta'}.$$

The critical area $A = A_{\mathrm{cr}}$ corresponds to $\theta_c = \pi$ giving

$$A_{\mathrm{cr}} = \pi \left[ \int \frac{\sigma_1(\theta)\, d\theta}{\pi - \theta} \right]^{-1}. \tag{4.5}$$

Note that the integral in (4.5) is not a principal value integral. Rather, it is an ordinary integral which has no singularities because $\theta = \pi$ lies outside of the support of $\sigma_1$.

Like the duality relation formula (4.5) can be checked in a number of exactly solvable cases. For example, QCD on a sphere corresponds to $\sigma_1(\theta) = \delta(\theta)$ so that $\int \frac{\sigma_1(\theta)\, d\theta}{\pi - \theta} = \frac{1}{\pi}$. Thus (4.5) yields $A_{\mathrm{cr}} = \pi^2$ reproducing the original result of Douglas and Kazakov.

The same technique can be used to obtain the transition area for a cylinder whenever we can evaluate the large $N$ limit of the corresponding Itzykson–Zuber integral (2.34) with the eigenvalue distributions $\alpha = \sigma_1$ and $\beta = \sigma_2$. This way we can generalize (4.5) to the case of a *flat* $\sigma_2$, that is,

$$\sigma_2(\theta) = \begin{cases} \dfrac{1}{2c_2}, & |\theta| \leq c_2 \\ 0 & \text{otherwise} \end{cases}$$

---

[25] Still, at $t = t_c$ the velocities *inside* of the support do not have to vanish.

Then the $c_2 \to 0$ limit would correspond to $\sigma_2(\theta) \to \delta(\theta)$.

For such $\sigma_2$ the eigenvalues $\theta_j^{(2)}$ are distrubuted uniformly over the interval $[-c_2, c_2]$:

$$\theta_j^{(2)} = \frac{2c_2}{N}\left(j - \frac{N}{2}\right) \qquad j = 1 \dots N$$

and the determinant in the Itzykson–Zuber formula (2.32) reduces to a Van der Monde determinant[26]

$$\det\left\|e^{2c_2 a_k(j-\frac{N}{2})}\right\| = e^{-Nc_2\sum_{k=1}^{N} a_k}\det\left\|\left(e^{2c_2 a_k}\right)^j\right\| = e^{-Nc_2\sum_{k=1}^{N} a_k}\,\Delta\!\left(e^{2c_2 a_k}\right)$$

$$= \exp\left[-Nc_2\sum_{k=1}^{N} a_k + \frac{1}{2}\sum_{j\neq k=1}^{N}\ln\left|e^{2c_2 a_j} - e^{2c_2 a_k}\right|\right].$$

Thus the large $N$ limit is easy to evaluate with the result that the action functional $S$ equals[27]

$$S = -\frac{1}{2t}\int \sigma_1(\theta)\,\theta^2\,d\theta - \frac{c_2}{t}\int \sigma_1(\theta)\,\theta\,d\theta$$
$$+ \frac{1}{2}\int\int \sigma_1(\theta)\,\sigma_1(\theta')\,\ln\left|e^{\frac{2c_2}{t}\theta} - e^{\frac{2c_2}{t}\theta'}\right|\,d\theta\,d\theta'. \tag{4.6}$$

Therefore the initial Hopf velocity solving the boundary problem (2.16) can be written down as

$$v_1(\theta) = \frac{\partial}{\partial\theta}\frac{\delta S}{\delta\sigma_1(\theta)} = -\frac{\theta + c_2}{t} + \frac{2c_2}{t}\int \frac{\sigma_1(\theta')\,d\theta'}{1 - e^{\frac{2c_2}{t}(\theta'-\theta)}}$$
$$= -\frac{\theta}{t} + \frac{c_2}{t}\int \frac{\sigma_1(\theta')\,d\theta'}{\tanh\left[\frac{c_2}{t}(\theta - \theta')\right]} \tag{4.7}$$

so that the initial function $f_0$ in (2.30) is

$$f_0(\theta) = -\frac{\theta}{t} + \frac{c_2}{t}\int \frac{\sigma_1(\theta')\,d\theta'}{\tanh\left[\frac{c_2}{t}(\theta - \theta')\right]}.$$

As before, the critical area $A_{\mathrm{cr}}$ can be found from

$$f_0(\theta_c = \pi)\big|_{t=A_{\mathrm{cr}}} = 0 \tag{4.8}$$

giving finally the formula

$$\int \frac{\sigma_1(\theta)\,d\theta}{\tanh\left[\frac{c_2}{A_{\mathrm{cr}}}(\pi - \theta)\right]} = \frac{\pi}{c_2}. \tag{4.9}$$

---

[26]   We keep in mind the correspondence $\theta_k^{(1)} \leftrightarrow a_k$, $\theta_j^{(2)} \leftrightarrow b_j$ and $\alpha(\theta) \leftrightarrow \sigma_1(\theta)$, $\beta(\theta) \leftrightarrow \sigma_2(\theta)$.

[27]   See footnote 24 for the explanation of $t$-dependence.

In the limit $c \to 0$ this equation reproduces (4.5). On the other hand if $\sigma_1$ also is a flat distribution, $\sigma_1(\theta) = \frac{1}{2c_1}$ for $|\theta| \le c_1$ then (4.9) gives us

$$\tanh\frac{\pi c_1}{A_{\mathrm{cr}}} \tanh\frac{\pi c_2}{A_{\mathrm{cr}}} = \tanh\frac{c_1 c_2}{A_{\mathrm{cr}}}. \tag{4.10}$$

## 5. Two-dimensional QCD on a vertex manifold.

Another nontrivial problem is the large $N$ QCD partition function on a vertex manifold (fig. 2), or pants diagram. In a sense, the cylinder and the "vertex" are the only nontrivial manifolds we have to consider. Any other two-dimensional surface can be constructed by gluing together an appropriate number of vertices and cylinders. Then, according to (2.19)-(2.22) the partition function on such composite manifold is just the product of partition functions for its constituents. Moreover, we can restrict ourselves to the limit when the area of the vertex manifold goes to zero. Indeed, we can create a vertex of any finite area just by attaching cylinders to a vertex of infinitely small area. In this case the partition function of QCD on a vertex becomes a certain functional of the three eigenvalue distributions for the boundary matrices[28] $U_{C_1}$, $U_{C_2}$ and $U_{C_3}$.

This functional should be somewhat similar to a delta-function. For one, in the case of a cylinder if $A \to 0$

$$\mathcal{Z}_N(U_1, U_2 | A) \to \sum_R \chi_R(U_1)\chi_R(U_2^\dagger) = \delta_{\mathrm{cl}}(U_1, U_2) \tag{5.1}$$

where $\delta_{\mathrm{cl}}$ is the delta-function on the set of conjugacy classes of the gauge group. That is to say, $\delta_{\mathrm{cl}}$ equates the sets of eigenvalues of $U_1$ and $U_2$. We can represent it in terms of the usual delta-function on a group manifold $\delta(U)$:

$$\delta_{\mathrm{cl}}(U_1, U_2) = \int dU\, \delta(U_1 U U_2^\dagger U^\dagger, I) \tag{5.2}$$

with $I$ the identity matrix.

It is useful to think of this function as analogous to the vertices of conventional field theory. There, with every vertex there is associated a delta-function of the total incoming momentum. Obviously, something similar should appear in our case as well. We expect that the structure of the partition function of QCD on the vertex manifold will have two ingredients. One is a selection condition (similar to the condition that the sum of all incoming momenta is zero) and the other ingredient is the actual value of the partition

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[28] Throughout this chapter these matrices will be denoted simply as $U_1$, $U_2$ and $U_3$.

function when this condition is satisfied (the counterpart of the vertex coefficient in field theory).

To find the selection condition we start with the exact expression for the partition function of $\mathrm{QCD}_2$ on the vertex manifold of a finite area $A$

$$\mathcal{Z}_N(U_1, U_2, U_3|A) = \sum_R \frac{\chi_R(U_1)\chi_R(U_2)\chi_R(U_3)}{d_R} e^{-\frac{A}{2N}C_2(R)}. \tag{5.3}$$

As $A \to 0$,

$$\mathcal{Z}_N \to \sum_R \frac{\chi_R(U_1)\chi_R(U_2)\chi_R(U_3)}{d_R} \equiv \mathcal{Z}_N(U_1, U_2, U_3).$$

Using the formulas

$$\int dV\, \chi_R(AVBV^\dagger) = \frac{1}{d_R}\chi_R(A)\chi_R(B)$$
$$\sum_R \chi_R(U)\chi_R(V^\dagger) = \delta_{\mathrm{cl}}(U, V) \tag{5.4}$$

we can rewrite this in the form analogous to (5.1):

$$\mathcal{Z}_N(U_1, U_2, U_3) = \sum_R \int dV_1\, \chi_R(V_1 U_1 V_1^\dagger U_2)\chi_R(U_3)$$
$$= \int dV_1\, dV_2\, \delta(V_1 U_1 V_1^\dagger V_2 U_2 V_2^\dagger U_3, I). \tag{5.5}$$

Since the characters in (5.3) are class functions on the group we can assume that $U_1$, $U_2$ and $U_3$ are diagonal matrices.

To evaluate the integral in (5.5) we need to find matrices $V_1$ and $V_2$ that satisfy the equation $V_1 U_1 V_1^\dagger V_2 U_2 V_2^\dagger U_3 = I$ and then compute the Jacobians necessary to integrate out the delta-function.

First, we rewrite (5.5) as

$$\mathcal{Z}_N(U_1, U_2, U_3) = \int dV_1\, dV_2\, \delta\big(V_1\big(U_1(V_1^\dagger V_2)U_2(V_1^\dagger V_2)^\dagger\big)V_1^\dagger, U_3^\dagger\big)$$
$$= \int dV_1\, dM\delta\big(V_1(U_1 M U_2 M^\dagger)V_1^\dagger, U_3^\dagger\big). \tag{5.6}$$

Now, we can perform the integration with respect to $V_1$, keeping $M$ fixed. The delta-function will pick up the specific value of $V_1 = V_1^{(0)}$ such that $V_1^{(0)}$ diagonalizes the matrix $Q(M) \equiv U_1 M U_2 M^\dagger$. Thus, after the integration over $V_1$ the delta-function in (5.6) reduces to a delta-function equating the eigenvalues of $Q(M)$ and $U_3^\dagger$. Indeed, if $V_1^{(0)}Q(M)V_1^{(0)\dagger} =$

$\Lambda$ where $\Lambda$ is diagonal then in the small vicinity of $V_1^{(0)}$ we have $V_1 = (I + v_1)V_1^{(0)}$ with some antihermitian $v_1$ and

$$\delta(V_1 Q(M) V_1^\dagger, U_3^\dagger) = \delta(\Lambda + [v_1, \Lambda], U_3^\dagger)$$

$$= \prod_{p<q} \delta(\text{Re}\,[v_1, \Lambda]_{pq}) \delta(\text{Im}\,[v_1, \Lambda]_{pq}) \prod_{p=1}^{N} \delta(\Lambda_p - (U_3^\dagger)_p)$$

$$= \prod_{p<q} \frac{\delta(\text{Re}\,(v_1)_{pq}) \delta(\text{Im}\,(v_1)_{pq})}{|\Lambda_p - \Lambda_q|^2} \prod_{p=1}^{N} \delta(\Lambda_p - (U_3^\dagger)_p)$$

where $\Lambda_p$ and $(U_3)_p$ are the eigenvalues of the (diagonal) matrices $\Lambda$ and $U_3$, respectively. Thus, up to a $\Lambda$-dependent coefficient $c(\Lambda)$,

$$\mathcal{Z}_N(U_1, U_2, U_3) \sim \int dM\, c(\Lambda(M)) \prod_{p=1}^{N} \delta(\Lambda_p(M) - (U_3)_p^*) \qquad (5.7)$$

where $\Lambda_p(M)$ are the eigenvalues of $Q(M) \equiv U_1 M U_2 M^\dagger$. To estimate this integral we should find $M = M_0$ such that $Q(M_0)$ has the same eigenvalues as $U_3^\dagger$ and compute the Jacobian $J(M) = \det \left\| \frac{\partial \Lambda_p(M)}{\partial M_{ab}} \right\|_{a,b=1}^{N}$. The partition function $\mathcal{Z}_N$ will be inversely proportional to this Jacobian.

If this Jacobian vanishes the partition function blows up. Intuitively, in this case the volume of integration over $dM$ which contributes to (5.7) is much larger than in the generic case. The most singular situation is when not only the Jacobian $J(M)$ bot the whole matrix $\frac{\partial \Lambda_p(M)}{\partial M_{ab}}$ vanishes. Then any small shift away from $M = M_0$ does not violate the condition imposed by the delta-function in (5.7) and the integration volume will be the largest. The condition on $U_1$, $U_2$ and $U_3$ under which this occurs will be an analogue of the condition $\sum_{i=1}^{n} p_i = 0$ imposed by the vertices in field theory.

To find this condition we must calculate the variation of eigenvalues $\delta\Lambda_p(M) = \Lambda_p(M + \delta M) - \Lambda_p(M)$ to the first order in $\delta M$ and require that such variation vanishes.

But $\Lambda_p(M)$ are the eigenvalues of the matrix $Q(M)$ with the variation

$$\delta Q(M) = U_1 \delta M U_2 M_0^\dagger - U_1 M_0 U_2 M_0^\dagger \delta M M_0^\dagger.$$

If

$$V_1^{(0)} Q(M_0) V_1^{(0)\dagger} = V_1^{(0)} U_1 M_0 U_2 M_0^\dagger V_1^{(0)\dagger} = \Lambda(M_0) \qquad (5.8)$$

is diagonal, then the variations of the diagonalizing matrices $V_1^{(0)}$ do not contribute to the change in the eigenvalues and, according to first order perturbation theory

$$\delta\Lambda_p(M) = \left(V_1^{(0)} \delta Q(M) V_1^{(0)\dagger}\right)_{pp}$$

$$= (V_1^{(0)} U_1 \delta M U_2 M_0^\dagger V_1^{(0)\dagger})_{pp} - (V_1^{(0)} U_1 M_0 U_2 M_0^\dagger \delta M M_0^\dagger V_1^{(0)\dagger})_{pp} \qquad (5.9)$$

$$= \left(V_1^{(0)} U_1 [\delta M M_0^\dagger, U_1 M_0 U_2 M_0^\dagger] V_1^{(0)\dagger}\right)_{pp}.$$

If we introduce the matrices $\mu = \delta M M_0^\dagger$, $\nu = V_1^{(0)} \mu V_1^{(0)\dagger}$ and $K_1 = V_1^{(0)} U_1 V_1^{(0)\dagger}$ and take into account that, due to (5.8) ,

$$M_0 U_2 M_0^\dagger = U_1^\dagger V_1^{(0)\dagger} \Lambda(M_0) V_1^{(0)}$$

we obtain the following representation for (5.9):

$$\delta \Lambda_p(M) = \left(K_1 \nu K_1^\dagger \Lambda(M_0)\right)_{pp} - \left(\Lambda(M_0)\nu\right)_{pp} = \left((K_1 \nu K_1^\dagger)_{pp} - \nu_{pp}\right)\Lambda_p(M_0) \qquad (5.10)$$

since $\Lambda(M_0)$ is a diagonal matrix.

The condition $\delta \Lambda_p(M) = 0$ means that $(K_1 \nu K_1^\dagger)_{pp} = \nu_{pp}$ for any $p$ and any antihermitian $\nu = -\nu^\dagger$. This is possible only if $K_1$ is diagonal, that is $(K_1)_{pr} = c_p \delta_{pr}$. Thus $V_1^{(0)} U_1 V_1^{(0)\dagger} = K_1$ where both $U_1$ and $K_1$ are diagonal matrices. This means that $V_1^{(0)}$ is a permutation matrix and $c_p$ are the same as the eigenvalues $(U_1)_p$ (maybe in a different order). In turn, this implies the diagonality of $U_1^\dagger V_1^{(0)\dagger} \Lambda(M_0) V_1^{(0)}$ and, because of (5.8), also the diagonality of $M_0 U_2 M_0^\dagger$. Since $U_2$ is diagonal, $M_0$ also is a permutation matrix, and so is $V_2^{(0)} = V_1^{(0)} M_0$. In short, $V_1^{(0)}$, $V_2^{(0)}$ and $M_0$ are some permutation matrices.

Going back to (5.5) we conclude that $(V_1^{(0)} U_1 V_1^{(0)\dagger})(V_2^{(0)} U_2 V_2^{(0)\dagger}) U_3 = I$. In other words, *when properly ordered the eigenvalues of $U_1$, $U_2$ and $U_3$ must give unity upon multiplication.*

This conclusion can be easily reformulated in terms of eigenvalue distributions $\sigma_1(\theta)$, $\sigma_2(\theta)$ and $\sigma_3(\theta)$. To do this, we introduce the "numbering functions" $\Sigma_j(\theta)$, $j = 1, 2, 3$, such that $d\Sigma_j(\theta)/d\theta = \sigma_j(\theta)$. These functions map the eigenvalue $\theta_i^{(j)}$ with number $i \in \{1, \ldots, N\}$ to the fraction $\frac{i}{N} \in [0, 1]$. If we now construct the inverse mappings $V_j(u)$ so that $\Sigma_j\big(V_j(u)\big) = u$ (no summation over $j$) then the multiplication condition for eigenvalues[29] $\lambda_i^{(1)} \lambda_i^{(2)} \lambda_i^{(3)} = 1$ is equivalent to

$$V_1(u) + V_2(u) + V_3(u) = 0. \qquad (5.11)$$

This is the selection condition imposed by the zero-area vertex for large $N$.

This discussion shows that the functions $V_j(u)$ are somewhat analogous to momenta in a conventional field theory. Moreover, using the formalism of subsection 2.3 it is possible to show that the velocities $v_j(\theta)$ in the collective field representation satisfy a similar constraint,

$$v_1\big(V_1(u)\big) = v_2\big(V_2(u)\big) = v_3\big(V_3(u)\big). \qquad (5.12)$$

---

[29] Here we implicitly presume that the eigenvalues have already been proprely ordered, so that in fact $U_1 U_2 U_3 = I$ is satisfied.

This constraint together with

$$\Sigma_1\big(V_1(u)\big) = \Sigma_2\big(V_2(u)\big) = \Sigma_3\big(V_3(u)\big)$$

has a suggestive form. Let us recall that the purpose of the zero-area vertices is to glue three cylinders together. The functions $V_j$ provide mappings between the boundaries of these cylinders which are similar to the mappings between different coordinate patches forming the atlas of a manifold. This may mean that the string theory of two-dimensional QCD is most naturally formulated in terms of functions, functionally inverse to the collective field variables $v$ and $\Sigma$.

Another application of (5.11) concerns the problem of the master field. The collective field theory gives us a way of evaluating the Wilson loops only without self-intersections. However, by joining a number of nonintersecting loops we can obtain an arbitrarily complex Wilson loop. Such joining is described by a vertex function we are discussing. However, to evaluate the composite Wilson loop one would need to know not merely the selection condition (5.11), but also the vertex function itself when this condition is satisfied[30]. This requires more subtle methods of calculation and is a separate subject that will be discussed elsewhere.

## 6. Acknowledgements.

We are indebted to M. Douglas and V. Kazakov for very helpful conversations.

## Appendix A. The change of variables in the collective field description of $\mathrm{QCD}_2$.

Our goal is to find the large $N$ limit of equation (2.11). To this end we set $\tilde{\mathcal{Z}}_N = \exp N^2 \tilde{F}_N$ and transform (2.11) into an equation for $\tilde{F}_N$ with the result

$$
\begin{aligned}
2\frac{\partial \tilde{F}_N}{\partial A} =& \frac{1}{N}\sum_{k=1}^{N}\frac{\partial^2 \tilde{F}_N}{\partial \theta_k^{(1)2}} + \frac{1}{N}\sum_{k=1}^{N}\left(N\frac{\partial \tilde{F}_N}{\partial \theta_k^{(1)}}\right)^2 \\
&+ \frac{2}{N}\sum_{k=1}^{N}U_k\left(N\frac{\partial \tilde{F}_N}{\partial \theta_k^{(1)}}\right) + \frac{1}{N^3}\sum_{k=1}^{N}\frac{1}{\mathcal{D}(\theta^{(1)})}\frac{\partial^2}{\partial \theta_k^{(1)2}}\mathcal{D}(\theta^{(1)})
\end{aligned}
\tag{A.1}
$$

where

$$
U_k \equiv \frac{1}{N}\frac{\partial}{\partial \theta_k^{(1)}}\ln \mathcal{D}(\theta^{(1)}) = \frac{1}{2N}\sum_{j\neq k}\cotan\frac{\theta_k^{(1)} - \theta_j^{(1)}}{2}.
\tag{A.2}
$$

---

[30]  This would be a counterpart of the vertex coefficient in field theory.

In the large $N$ limit all partial derivatives can be replaced by the derivatives with respect to the eigenvalue densities,

$$N\frac{\partial \tilde{F}_N}{\partial \theta_k^{(1)}} = \frac{\partial}{\partial \theta}\frac{\delta \tilde{F}}{\delta \sigma_1(\theta)}\bigg|_{\theta=\theta_k}$$

and all sums — by the integrals

$$\frac{1}{N}\sum_{k=1}^{N} \to \int_0^{2\pi} \sigma_1(\theta)\,d\theta.$$

At the same time

$$\frac{\partial^2 \tilde{F}_N}{\partial \theta_k^{(1)2}} \sim \mathcal{O}\Big(\frac{1}{N}\Big)$$

and can be neglected. As to the last term in (A.1),

$$\frac{1}{N^3}\sum_{k=1}^{N}\frac{1}{\mathcal{D}(\theta^{(1)})}\frac{\partial^2}{\partial \theta_k^{(1)2}}\mathcal{D}(\theta^{(1)}) = \frac{1}{N^2}\sum_{k=1}^{N}\frac{1}{\mathcal{D}(\theta^{(1)})}\frac{\partial}{\partial \theta_k^{(1)}}\Big[U_k \mathcal{D}(\theta^{(1)})\Big]$$

$$= \frac{1}{N^2}\sum_{k=1}^{N}\Big[\frac{\partial U_k}{\partial \theta^{(1)}} + U_k^2\Big] = \frac{1}{N^3}\sum_{\substack{j,k \\ j\neq k}}\frac{1}{4\sin^2\frac{\theta_k^{(1)}-\theta_j^{(1)}}{2}} + \frac{1}{N}\sum_{k=1}^{N}U_k^2.$$

When $N\to\infty$

$$U_k \to U(\theta) = \frac{1}{2}\int \cot\Big(\frac{\theta_k-\theta'}{2}\Big)\sigma_1(\theta')\,d\theta'$$

and

$$\frac{1}{N}\sum_{k=1}^{N}U_k^2 = \int_0^{2\pi} U^2(\theta)\,\sigma_1(\theta)\,d\theta.$$

As to the term

$$\frac{1}{N^3}\sum_{j\neq k}\frac{1}{4\sin^2\frac{\theta_k^{(1)}-\theta_j^{(1)}}{2}}$$

it might appear that it is $\mathcal{O}(1/N)$, since it contains only two summations (each of which naively contributes a factor of $N$) divided by $N^3$. However, due to the singularity at $\theta_k = \theta_j$ this term is in fact not negligible. Its $\mathcal{O}(1)$ contribution comes from the regions where $|k-j| \ll N$ so that $\theta_k^{(1)} - \theta_j^{(1)} \simeq (k-j)/(N\sigma_1(\theta_k^{(1)}))$ and

$$\frac{1}{N^3}\sum_{\substack{j,k \\ j\neq k}}\frac{1}{4\sin^2\frac{\theta_k^{(1)}-\theta_j^{(1)}}{2}} = \frac{1}{N^3}\sum_{\substack{j,k \\ j\neq k}}\frac{1}{(\theta_k^{(1)}-\theta_j^{(1)})^2} = \frac{1}{N^3}\sum_{k=1}^{N}\Big[\sum_{\substack{j=1 \\ j\neq k}}^{N}\frac{N^2\sigma_1^2(\theta_k^{(1)})}{(j-k)^2}\Big]$$

$$= \frac{1}{N}\sum_{k=1}^{N}\frac{\pi^2}{3}\sigma_1^2(\theta_k^{(1)}) = \frac{\pi^2}{3}\int_0^{2\pi}\sigma_1^3(\theta)\,d\theta$$

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where we used the identity

$$\sum_{j \neq k} \frac{1}{(j-k)^2} = \frac{\pi^2}{3}.$$

Combining the pieces we obtain

$$2\frac{\partial \tilde{F}}{\partial A} = \int\limits_0^{2\pi} \sigma_1(\theta)\, d\theta \left\{ \left(\frac{\partial}{\partial \theta} \frac{\delta \tilde{F}}{\delta \sigma_1(\theta)}\right)^2 + 2U(\theta)\frac{\partial}{\partial \theta} \frac{\delta \tilde{F}}{\delta \sigma_1(\theta)} + U^2(\theta) + \frac{\pi^2}{3}\sigma_1^2(\theta) \right\}. \tag{A.3}$$

If we now introduce a new functional $S\big[\sigma_1(\theta), \sigma_2(\theta)\big|A\big]$ according to (2.12) then

$$\frac{\partial}{\partial \theta} \frac{\delta \tilde{F}}{\delta \sigma_1(\theta)} + U(\theta) = \frac{\partial}{\partial \theta} \frac{\delta S}{\delta \sigma_1(\theta)}$$

so that (A.3) entails (2.13).

Let us emphasize an important aspect of this derivation. In making the transition from (A.1) to (A.3) we relied on the fact that the large $N$ limit $\tilde{F} = \lim_{N \to \infty} \tilde{F}_N$ exists. The same would not be true if we had attempted a shortcut, trying to replace $\mathcal{D}(\theta^{(1)})\tilde{\mathcal{Z}}_N$ by $\exp(N^2 \tilde{G}_N)$ in (2.11). In fact, the functional $\mathcal{D}(\theta^{(1)})\tilde{\mathcal{Z}}_N$ is not positive definite and therefore $\tilde{G}_N$ does not have a well defined large $N$ limit. The functional $S$ introduced in (2.12) is related to $\tilde{\mathcal{Z}}_N$ by $\exp(N^2 S) = |\mathcal{D}(\theta^{(1)})||\mathcal{D}(\theta^{(2)})|\tilde{\mathcal{Z}}_N$, the absolute value signs being essential. In fact, it is the difference between $\mathcal{D}(\theta^{(1)})$ and $|\mathcal{D}(\theta^{(1)})|$ that gives rise to the interaction term $\frac{\pi^2}{3}\int \sigma_1^3(\theta)\, d\theta$ in (2.13).


### Appendix B. The large $N$ limit of the Itzykson–Zuber integral.

The large $N$ limit of the Itzykson–Zuber integral (2.32) can be studied using essentially the same technique that we used in appendix A to obtain the collective field theory of QCD. First, one represents $I_N$ in the form

$$I_N(A, B) = \frac{\det \|e^{N a_k b_j}\|}{\Delta(a_k)\Delta(b_j)} = e^{\frac{N}{2}\left[\sum_{k=1}^N a_k^2 + \sum_{k=1}^N b_j^2\right]} \frac{\det \|e^{-\frac{N}{2}(a_k - b_j)^2}\|}{\Delta(a_k)\Delta(b_j)}. \tag{B.1}$$

Then it is easy to check directly that the quantity

$$J_N(t|A, B) = \frac{1}{t^{\frac{N^2}{2}}} \frac{\det \|e^{-\frac{N}{2t}(a_k - b_j)^2}\|}{\Delta(a_k)\Delta(b_j)} \tag{B.2}$$

satisfies the partial differential equation

$$2N\frac{\partial J_N}{\partial t} = \frac{1}{\Delta(a)} \sum_{i=1}^N \frac{\partial^2}{\partial a_i^2}\big[\Delta(a)\, J_N\big]. \tag{B.3}$$

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Once $J_N$ is known, $I_N$ can be retrieved as

$$I_N = J_N(t=1) \exp\left[\frac{N}{2}\left(\sum_{k=1}^{N} a_k^2 + \sum_{k=1}^{N} b_j^2\right)\right].$$

Equation (B.3) which is analogous to (2.11) can be treated by methods described in appendix A. The only difference is that the quantity $U_k$ is given not by (A.2) but by

$$U_k \equiv \frac{1}{N}\frac{\partial}{\partial a_k} \ln \Delta(a) = \frac{1}{N}\sum_{j\neq k} \frac{1}{a_k - a_j} \tag{B.4}$$

with the large $N$ limit

$$U(a) = \int \frac{\alpha(a')\, da'}{a - a'}$$

where $\alpha(a)$ is the density of the eigenvalues $\{a_k\}$. Performing the transformations described in appendix A one deduces that the functional

$$\begin{aligned}
S[t|\alpha,\beta] = \lim_{N\to\infty}\left\{\frac{1}{N^2}\ln J_N(t|A,B)\right\} &+ \frac{1}{2}\int \alpha(a)\,\alpha(a')\,\ln|a-a'|\,da\,da' \\
&+ \frac{1}{2}\int \beta(b)\,\beta(b')\,\ln|b-b'|\,db\,db'
\end{aligned} \tag{B.5}$$

satisfies the following differential equation

$$\frac{\partial S}{\partial t} = \frac{1}{2}\int_{-\infty}^{+\infty} \alpha(a)\left[\left(\frac{\partial}{\partial a}\frac{\delta S}{\delta\alpha(a)}\right)^2 - \frac{\pi^2}{3}\alpha^2(a)\right]da. \tag{B.6}$$

Like (2.13) this is the Hamilton–Jacobi equation for the dynamical system with the Hamiltonian

$$H\big[\rho(a),\Pi(a)\big] = \frac{1}{2}\int_{-\infty}^{+\infty} \rho(a)\left[\left(\frac{\partial\Pi}{\partial a}\right)^2 - \frac{\pi^2}{3}\rho^2(a)\right]da \tag{B.7}$$

However, as opposed to (2.13), in this dynamical system $\rho(a)$ is the distribution on an infinite real line rather than on a circle. In fact, it is the noncompactness of support of eigenvalue densities that distinguishes the Itzykson–Zuber integral from the large $N$ QCD.

Equation (B.6) is very hard, if not impossible, to solve in general. Fortunately, however, it is easy to find its particular solution which describes the large $N$ limit of the Itzykson–Zuber integral [17]. This solution is given by the action of dynamical system (B.7) along its classical trajectory connecting the densities $\rho(a) = \alpha(a)$ and $\rho(b) = \beta(b)$

within time $t$. It is not difficult to show that such quantity (which is known in classical mechanics as the principal Hamilton function) does indeed satisfy (B.6). Moreover, the variational derivatives at the end of the trajectory equal the corresponding canonical momenta:

$$\frac{\delta S}{\delta \alpha(a)} = \Pi(a, t = 0), \qquad \frac{\delta S}{\delta \beta(b)} = -\Pi(b, t = 1).$$

To find the above mentioned trajectory we must solve the equations of motion which follow from the Hamiltonian (B.7). As in (2.17), (2.29) these can be conveniently transformed into a single equation for the quantity $f(t, a) = \frac{\partial \Pi(a)}{\partial a} + i\pi\rho(a)$:

$$\frac{\partial f}{\partial t} + f\frac{\partial f}{\partial a} = 0. \tag{B.8}$$

Then

$$\frac{\partial}{\partial a}\frac{\delta S}{\delta \alpha(a)} = \frac{\partial \Pi(a, t = 0)}{\partial a} = \operatorname{Re} f(t = 1, a).$$

Now we are able to derive the constraints (2.36). We notice that the general solution of (B.8) can be written down in the parametric form

$$\begin{cases} x = R(\xi) + F(\xi)\, t \\ f(x, t) = F(\xi) \end{cases} \tag{B.9}$$

where $R(\xi)$ and $F(\xi)$ are some functions of the formal parameter $\xi$. These functions should be determined from the initial conditions. The conditions to be imposed in our case are $\operatorname{Im} f(t = 1, a) = \pi\alpha(a)$ and $\operatorname{Im} f(t = 0, b) = \pi\beta(b)$. If we introduce the two analytic functions $G_+(x)$ and $G_-(x)$ according to

$$\begin{cases} G_+(x) = x + f(t = 0, x) \\ G_-(x) = x - f(t = 1, x) \end{cases} \tag{B.10}$$

then $\operatorname{Im} G_+(x) = \pi\alpha(x)$ and $\operatorname{Im} G_-(x) = -\pi\beta(x)$. In addition, from (B.9) we deduce

$$G_+(x) = x + [F \circ R^{-1}](x) = \big[(F + R) \circ R^{-1}\big](x)$$

as well as

$$G_-(x) = x - \big[F \circ (F + R)^{-1}\big](x) = \big[R \circ (F + R)^{-1}\big](x)$$

where $\circ$ denotes functional composition. Therefore,

$$G_+\big(G_-(x)\big) = G_-\big(G_+(x)\big) = x$$

completing the proof of (2.36).

## Appendix C. Wilson loops on a plane and the Hopf equation.

In this appendix we will review the derivation of Wilson loops on a plane which uses the method of loop equations [8][19]. Using this example we will be able to see the close connection existing between the Hopf equation (2.29) and the Kazakov–Kostov equations for the two-dimensional QCD. In addition we will derive the explicit expressions (2.54) for Wilson loops on a plane.
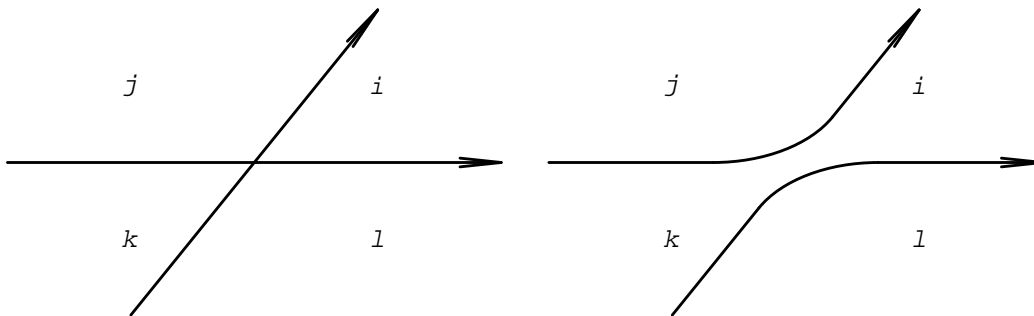


**Fig. 6:** The reconnection of a Wison loop at the self-intersection point employed in the two-dimensional loop equations.

Generally, the loop equations relate the value of a self-intersecting Wilson loop $W$ (fig.7) to the values of Wilson loops with smaller number of self-intersections. In particular, if we "reconnect" the self-intersection on the left in fig. 6 to obtain the contour shown on the right then the Wilson loop $W$ for the original contour satisfies the Kazakov–Kostov equation [8]

$$\hat{R}W = W'W''  \tag{C.1}$$

where the operator $\hat{R}$ is given by[31]

$$\hat{R} = \frac{\partial}{\partial S_k} + \frac{\partial}{\partial S_i} - \frac{\partial}{\partial S_l} - \frac{\partial}{\partial S_j}.  \tag{C.2}$$

In this formula $S_\alpha$ is the area of the window marked by the subscript $\alpha \in \{i, j, k, l\}$.

---

[31] We remember that in the two-dimensional QCD the Wilson loops depend only on the areas bounded by the contour but not on the contour shape. This follows from the invariance of $\mathrm{QCD}_2$ with respect to the group of area preserving diffeomorphisms.
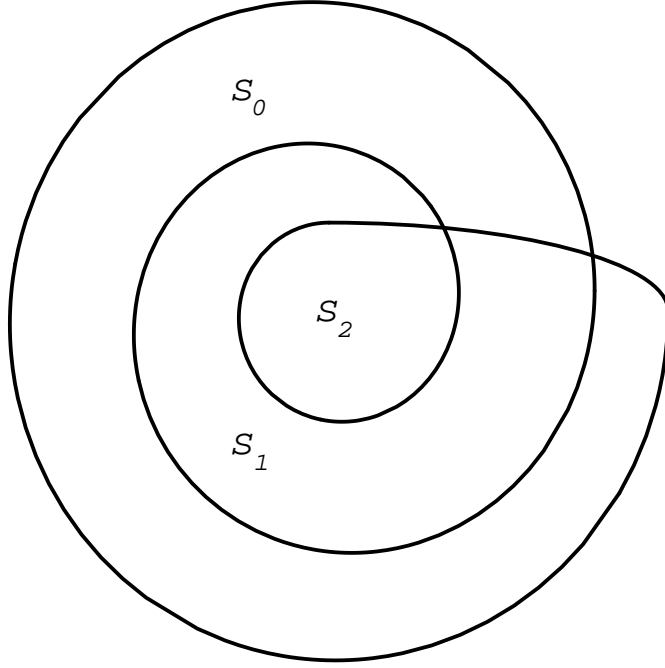
**Fig. 7:** A Wilson loop with two self-intersections (this corresponds to the winding number equal to 3). The areas of domains bounded by the loop are denoted as $S_0, S_1, S_2$, counting from the outside.

Below we will be interested in the loops which wind several times around the circle of area $A$. In fig. 7 this would correspond to a situation when $S_0, S_1, \ldots, S_{n-1} \to 0$, $S_n \to A$. In this case the Wilson loop averages $W_n(A)$ depend only on one variable $A$ and, due to (C.1), satisfy the equations

$$
\begin{aligned}
\left(2\frac{\partial}{\partial S_0} - \frac{\partial}{\partial S_1}\right)W_n &= W_{n-1}W_0 \\
\left(2\frac{\partial}{\partial S_1} - \frac{\partial}{\partial S_0} - \frac{\partial}{\partial S_2}\right)W_n &= W_{n-2}W_1 \\
&\vdots \\
\left(2\frac{\partial}{\partial S_k} - \frac{\partial}{\partial S_{k-1}} - \frac{\partial}{\partial S_{k+1}}\right)W_n &= W_{n-k-1}W_k \\
&\vdots
\end{aligned}
\tag{C.3}
$$

These equations can be simplified if we introduce a new set of variables $A_n = S_n$, $A_{n-1} = A_n + S_{n-1}, \ldots, A_k = A_{k+1} + s_k, \ldots$. Then $\partial/\partial A_0 = \partial/\partial S_0$ and $\partial/\partial A_k = \partial/\partial S_k - \partial/\partial S_{k+1}$

so that from (C.3) we obtain

$$\left(\frac{\partial}{\partial A_k} - \frac{\partial}{\partial A_{k+1}}\right) W_n = W_{n-k-1} W_k,$$

and, consequently,

$$\left(\frac{\partial}{\partial A_0} - \frac{\partial}{\partial A_k}\right) W_n(A_0, A_1, \ldots, A_n) = \sum_{l=0}^{k-1} W_{n-k-1}(A) W_k(A). \qquad (C.4)$$

Now, if $S_0, S_1, \ldots, S_{n-1} \to 0$ then $A_0 = A_1 = \ldots = A_n = A$ and

$$\frac{dW_n(A)}{dA} = \sum_{k=0}^{n} \left[ \frac{\partial}{\partial A_k} W_n(A_0, A_1, \ldots, A_n) \right]\Bigg|_{\text{all}A_k = A}. \qquad (C.5)$$

Let us also recall that the dependence of $W_n$ on the external area $A_0$ (or, the same, $S_0$) is always exponential, $W_n \sim e^{-A_0/2}$ so that

$$\frac{\partial W_n}{\partial A_0} = -\frac{1}{2} W_n.$$

This, together with (C.4) and (C.5) allows us to write down a recursion relation for $W_n$:

$$-\frac{dW_n}{dA} = \frac{n+1}{2} W_n + \sum_{l=0}^{n-1} (n-l) W_l W_{n-l-1}. \qquad (C.6)$$

Its solution has the general form

$$W_n(A) = P_n(A) e^{-\frac{n+1}{2} A}$$

where $P_n(A)$ is a polynomial in $A$ of degree $n$. It is easy to show by induction that these polynomials are given by (2.55) yielding the formula (2.54) [19].

Our goal here is different. We would like to see how this formalism corresponds with the collective field description of QCD and, particularly, with the Hopf equation (2.29).

In fact, the recursion relation (C.6) resembles the Fourier transform of the Hopf equation. Indeed, if we set

$$\psi(A, t) \equiv \sum_{n=0}^{\infty} W_n(A) e^{(n+1)(\frac{A}{2} + \zeta)}$$

then (C.6) entails

$$\frac{\partial \psi}{\partial A} + \psi \frac{\partial \psi}{\partial \zeta} = 0.$$

However, $\psi$ is not the same as the collective field function $f$ because by construction $\psi$ is real while $f$ has an imaginary part, $\operatorname{Im} f(t,\theta) = \pi\sigma_*(t,\theta)$. But if we put $\zeta = -i\theta - \frac{A}{2}$ then

$$\operatorname{Re}\psi\left(A, -i\theta - \frac{A}{2}\right) = \sum_{n=0}^{\infty} W_n(A) \cos(n+1)\theta = \pi\sigma_*(t,\theta) - \frac{1}{2}$$

where we have used the relation between $W_n$ and $\sigma_*$

$$W_n(C) = \int_0^{2\pi} \sigma_*(t = A_1, \theta)\, e^{in\theta}\, d\theta$$

(see (2.23)).

Thus the function

$$f(A,\theta) = i\left[\psi\left(A, -i\theta - \frac{A}{2}\right) + \frac{1}{2}\right] \tag{C.7}$$

has the correct imaginary part, $\operatorname{Im} f(A,\theta) = \pi\sigma_*(A,\theta)$. Moreover, since

$$\left(\frac{\partial}{\partial\zeta}\right)_A = i\left(\frac{\partial}{\partial\theta}\right)_A, \quad \left(\frac{\partial}{\partial A}\right)_\zeta = \left(\frac{\partial}{\partial A}\right)_\theta + \frac{i}{2}\left(\frac{\partial}{\partial\theta}\right)_A$$

the Hopf equation for $\psi$ transforms into the Hopf equation for $f$:

$$\left(\frac{\partial f}{\partial A}\right)_\theta + f\left(\frac{\partial f}{\partial\theta}\right)_A = 0. \tag{C.8}$$

Unlike $\psi$ the function $f$, which has both real and imagnary parts, is precisely the function used in the collective field theory. Thus, the formula (C.7) establishes the agreement between the Hopf equation and the loop equations of two-dimensional QCD.

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